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FOUNDATIONS

Popper, K. R. Functional logic without axioms or primitive rules of inference. Nederl. Akad. Wetensch., Proc. **50**, 1214–1224 = Indagationes Math. **9**, 561–571 (1947).

In an earlier paper [Mind **56**, 193–235 (1947); these Rev. **9**, 130] the author proposed a formulation for logical calculus in which the fundamental relation is one of deducibility expressed by "$a_1, \cdots, a_n/b$." This relation is similar to one postulated by P. Hertz [Math. Ann. **101**, 457–514 (1929)] and used by Gentzen [Math. Z. **39**, 176–210, 405–431 (1934)] in his system *LJ*. Popper proposed in that paper two "bases" for his formulation, called basis I and basis II, respectively. Basis I is similar to that of Hertz; its rules *Rg* and *Tg* are the rules stated, in that order, in the cited review. Basis II postulates *Tg* for the case $m=1$ only, but requires that there be an operation of conjunction, expressed by "$a \wedge b$" and a rule *Cg*: $a_1, \cdots, a_n/b \wedge c \rightleftarrows a_1, \cdots, a_n/b$ & $a_1, \cdots, a_n/c$. His contention is that basis I can be derived from basis II. The operations formalizing the logical connectives are introduced on basis I by rules which have, in interpretation at least, the character of definitions. In the present paper these "metalinguistic inferential definitions" are restated so as to incorporate in them the rules of basis II. Of course the "definitions" are not definitions in the usual sense, and the device is at best only a technical subterfuge. But any interest it may have is destroyed by the fact that the contention above mentioned is false. Counterexample: let the $a_1, \cdots, a_n, b$ be rational integers, $b \wedge c$ be the lesser of b and c, and let "$a_1, \cdots, a_n/b$" mean that the minimum of $a_1, \cdots, a_n$ is less than or equal to $b+n-1$; then the rules of basis II are satisfied, but *Tg* fails for $a=b_1=b_2=1$, $c=0$.

In view of this and other errors in the previous paper, conclusions based on it must be received with caution. Presumably the author's ideas can be carried through, at least in principle (the chief doubts relate to variables and quantifiers), for basis I. If the object is to simplify the rule *Tg*, Gentzen's "Hauptsatz" goes further in this direction than the author's basis II. The paper closes with a statement of possible definitions for modal functions. These require the use of a disjunction in the "metalanguage."

H. B. Curry (State College, Pa.).

Markov, A. The impossibility of certain algorithms in the theory of associative systems. II. Doklady Akad. Nauk SSSR (N.S.) **58**, 353–356 (1947). (Russian)

The first part of the paper is concerned with sharpening the author's theorems 1 and 2 [same Doklady (N.S.) **55**, 583–586 (1947); these Rev. **8**, 558] on the unsolvability of the word problem in associative systems [demi-groups, Thue systems], and on the unsolvability of the division problem ("has $XQ \leftrightarrow R$ a solution for given Q and R?"), respectively. (Notations: $X \vdash Y$ means that word X may be replaced by word Y, $X \perp Y$ means $X \vdash Y$ and $Y \vdash X$, $X \leftrightarrow Y$ means that $PXQ \perp PYQ$ for any P and Q.) (A) A system may have a solvable word problem and solvable left-division problem, but unsolvable right-division problem, $XH_1 \leftrightarrow R$; and here H_1 can be a fixed word. (B) An associative system is given having no decision algorithm for $X \leftrightarrow 0$ (the empty word). Proofs rest on E. L. Post's unsolvable word problem under a set of one-way substitutions $G_iP \vdash PG_i'$ $(i=1, 2, \cdots, m)$, G_i and G_i' fixed, P arbitrary [Amer. J. Math. **65**, 197–215 (1943); these Rev. **4**, 209].

In the second part of the paper a new step is taken by the construction of a class of associative systems with unsolvable word problem, having only a moderate number of defining relations which can easily be written out without abbreviation. In examples hitherto constructed the relations contain in effect a specification of the class of recursive functions and are correspondingly complicated. From the sparse indications given in the present paper, it appears that the complications are transferred from the relations to the class of words within the system for which the equivalence problem is posed. The main theorem is as follows. The alphabet consists of (a, b, c, d, e). Let V be an arbitrary word in the letters a and b, and U any word in a, b, c, d beginning and ending with d, in which every two c's are separated by at least one d. Then there is no algorithm for deciding whether UeV goes into Uea by means of the single rule (α) $PdQcRdSeQT \perp PdQcRdSeTR$, where P, S, T are arbitrary words in a, b, c, d, e; Q and R arbitrary words in a, b. The following two results are to be used in the proof. (1) In the problem of Post referred to above the one-way substitutions $G_iP \vdash PG_i'$ may be replaced by symmetrical ones $K_iP \perp PK_i'$. [Cf. lemma II of Post, J. Symbolic Logic **12**, 1–11 (1947); these Rev. **8**, 558.] (2) If L_i, L_i' $(i=1, \cdots, q)$, V, W are words in a, b, a necessary and sufficient condition that V go into W by means of $L_iP \perp PL_i'$ is that $dL_1cL_1' \cdots dL_qcL_q'deV$ goes into $dL_1cL_1' \cdots dL_qcL_q'deW$ by means of (α). An associative system C_8 on 13 letters is given, with 33 defining relations $X \leftrightarrow Y$, where no X or Y contains more than three letters, which can be shewn by the main theorem (presumably using the examples constructed in the first part of the paper) to have an undecidable word problem; in a slightly more complicated system C_6 the problem $X \leftrightarrow H_0$ is undecidable, where H_0 is a fixed 3-letter word. No details of the proof are given.

M. H. A. Newman (Manchester).

Tartakovskiĭ, V. On the process of extinction. Doklady Akad. Nauk SSSR (N.S.) **58**, 1605–1608 (1947). (Russian)

The author studies the word problem for groups $\mathfrak{G}$ with a finite number of generators $S_1, \cdots, S_m$ and relations

$$S_1^{n_1}=1, S_2^{n_2}=1, \cdots, S_m^{n_m}=1, \tag{1}$$

$$f_1(S)=1, f_2(S)=1, \cdots, f_k(S)=1; \tag{2}$$

the expressions $f_i(S)$ indicate words formed from the generators $S_1, \cdots, S_m$. Identity of two words φ_1 and φ_2 is indicated by the sign $\approx$; words φ_1 and φ_2 which become identical upon application of the identities (1) are called freely equal $(\varphi_1 \equiv \varphi_2)$. For two words φ_1 and φ_2 which are equal as ele-

ments of $\mathfrak{G}$, the expression $\varphi_1=\varphi_2$ is employed. Let $V'(n)$ be the set of all integers $x\neq 0$ such that $-n/2\leqq x<n/2$. A word (3) $S_{i_1}^{\alpha_1}\cdots S_{i_\nu}^{\alpha_\nu}$ is said to be reduced if $i_\rho\neq i_{\rho+1}$ $(\rho=1,2,\cdots,\nu-1)$ and $\alpha_\rho\varepsilon V'(n_\rho)$ $(\rho=1,\cdots,\nu)$. The vacuous word, where $\nu=0$, and denoted by the symbol 1, is said to be reduced also. The product of two words, $S_{i_1}^{\alpha_1}\cdots S_{i_k}^{\alpha_k}\cdot S_{j_1}^{\beta_1}\cdots S_{j_\sigma}^{\beta_\sigma}$ is said to be meshed if $i_k=j_1$; otherwise, unmeshed. A certain ordering is introduced into the set of reduced words. We have $S_{i_1}^{\alpha_1}\cdots S_{i_\nu}^{\alpha_\nu}>S_{j_1}^{\beta_1}\cdots S_{j_\sigma}^{\beta_\sigma}$ if $\nu>\sigma$, and for words of the same length, the ordering is antilexicographic. A word $\varphi(S)$ such that $\varphi(S)=1$ is called a Dyck word. A cyclic word is a word in which a new relation of equality is defined: two words $S_{i_1}^{\alpha_1}\cdots S_{i_\nu}^{\alpha_\nu}$ and $S_{i_{k+1}}^{\alpha_{k+1}}\cdots S_{i_\nu}^{\alpha_\nu}S_{i_1}^{\alpha_1}\cdots S_{i_k}^{\alpha_k}$ differing only by a cyclic permutation are to be identified. A cyclic word is denoted by the symbol $|\varphi(S)|$.

The region of extinction of the group $\mathfrak{G}$, denoted by the symbol $\pi(\mathfrak{G})$, is the set of all reduced words $\varphi(S)$ of $\mathfrak{G}$ such that $\varphi(S)=\psi(S)$ for some reduced word $\psi<\varphi$. A cyclic word may be made into a linear (i.e., ordinary) word by splitting at any point, and a linear word may be obviously made into a cyclic word. The set of all words $x\cdot R\cdot v$, where R is fixed, and x and v are subject only to the restriction that the products $x\cdot R$ and $R\cdot v$ are unmeshed, is called the ray with kernel R. If all the words of the ray lie in $\pi(\mathfrak{G})$, the ray is called a ray of extinction. Let $R(S)$ be a Dyck word split into two words $R_1(S)$ and $R_2(S)$. Then $R_2(S)$ is the inverse of $R_1(S)$ in G; and if $R_1(S)>R_2^{-1}(S)$, where $R_2^{-1}(S)$ is the inverse of $R_2(S)$ with the same factors as $R_2(S)$, written in inverse order, and with exponents lying in the appropriate $V'(n)$, then the ray $x\cdot R_1(S)\cdot v$ is called a left ray of extinction of $R(S)$. All such rays unite to form the cluster of extinction for $R(S)$. For a cyclic Dyck word $|R(S)|$, left rays and clusters are defined for arbitrary linear words formed from $|R(S)|$, and the set of all such clusters is called the star of extinction for $|R(S)|$.

The composition $R*S$ of two cyclic words $|R|$ and $|S|$ is defined by (1) splitting $|R|$ and $|S|$ into linear words, (2) multiplying these two words, (3) forming a cyclic word from this product. Given a set $A_1,\cdots,A_q$ of words, consider the set of all compositions $A_{i_1}*(A_{i_2}*(\cdots(A_{i_{r-1}}*A_{i_r})\cdots))$, where $1\leqq i_1<i_2<\cdots<i_r\leqq q$. The set of all stars of extinction of such words is called the zone of extinction of the set of words $A_1,\cdots,A_q$. Theorem: Two sets of words $A_1,\cdots,A_q$ and $B_1,\cdots,B_q$ have the same zones of extinction if the B_i and the A_i are Dyck words and $A_i\equiv B_i$ $(i=1,2,\cdots,q)$. A word is said to be simple if it is formed by the process of composition from powers of the words in relations (2). Theorem: $\pi(\mathfrak{G})$ is the union of the stars of extinction for all simple Dyck words of the group G. No proofs are given. *E. Hewitt* (Chicago, Ill.).

Post, Emil L. Recursively enumerable sets of positive integers and their decision problems. Revista Mat. Hisp.-Amer. (4) **7**, 187–229 (1947). (Spanish)

Translation of a paper in Bull. Amer. Math. Soc. **50**, 284–316 (1944); these Rev. **6**, 29.

van Dantzig, D. On the principles of intuitionistic and affirmative mathematics. II. Nederl. Akad. Wetensch., Proc. **50**, 1092–1103 = Indagationes Math. **9**, 506–517 (1947).

After explaining the deficiencies of the weak constructive interpretation of statements of part I [same Proc. **50**, 918–929 = Indagationes Math. **9**, 429–440 (1947); these Rev. **9**, 221], the author sketches a theory of real numbers from the point of view of what he has earlier called "affirmative mathematics." The work could be formalized in a predicate calculus obtained from the intuitionistic one by deletion of all principles involving the negation symbol or the existential quantifier. The author anticipates a future treatment avoiding the disjunction symbol. The arithmetic of natural numbers is developed from an induction schema, axioms for equality and Peano axioms, one of the latter in the affirmative form: for natural numbers x and y, $x+y=0\supset y=0$. Assuming a development of rationals, the real numbers are introduced by a function which gives essentially the nth rational approximation of the number in binary expansion. The relation $<$ is not definable for real numbers, but is "dispersed" into a class of relations holding between approximating rationals. Again because of the absence of the existential quantifier the limit operation is defined separately for each ϵ-function. The author concludes by observing that, in the formalization of mathematics, intuitively distinct situations are identified; hence we must expect the need for new formal systems to reflect possible refinements in analysis. *D. Nelson* (Washington, D. C.).

Beth, E. W. Semantical considerations on intuitionistic mathematics. Nederl. Akad. Wetensch., Proc. **50**, 1246–1251 = Indagationes Math. **9**, 572–577 (1947).

The paper contains a brief general description of semantical method and some of its important results for foundations of classical mathematics and raises the general question of the application of such methods to intuitionistic mathematics. It is observed that the justification of logical rules of inference (which the intuitionist holds necessary) might, at least for some systems of arithmetic, be accomplished by a semantical method assuming an intuitive construction of the natural numbers. The paper considers Brouwer's definition of a spread law, called "Menge" in Brouwer's earlier accounts [Verh. Akad. Wetensch. Amsterdam. Sect. I. **12**, no. 5 (1918)]. The author presents a method of expressing a spread law in symbolic form involving two sequences of functions. Supposing that these functions are to be defined in some recursive fashion, he conjectures that a limitation to any specific class of recursions would result in an incomplete formal representation of the spread laws.

D. Nelson (Washington, D. C.).

de Bengy Puyvallée, Renaud. Sur les règles de composabilité dans la logique de la mathématique intuitioniste sans négation. C. R. Acad. Sci. Paris **226**, 454–456 (1948).

The author continues his investigation of systems of logic in which it is not always possible to form the conjunction of two sentences (this formal peculiarity was originally suggested by certain characteristics of quantum theory). The paper is written in the author's usual style: i.e., symbolism is introduced which is not clearly explained and axioms are given whose significance is not precisely stated.

J. C. C. McKinsey (Santa Monica, Calif.).

***Kattsoff, Louis O. A Philosophy of Mathematics.** The Iowa State College Press, Ames, Iowa, 1948. ix+266 pp. $5.00.

This book provides an introduction to the basic problems of mathematics and presents some of the implications of these problems and their proposed solutions for philosophy in general. It is intended for advanced undergraduates.

Basic concepts. Distinction is made between the metamathematical and epistemological problems of mathematics, and between the critical and speculative aspects of philosophy. Eleven definitions of mathematics are listed and analyzed by types. The intuitionist, empirical, conventionalistic, logistic and formalistic positions are described and compared, and the "object" of mathematics is stated to be "those properties of entities in virtue of which they are phenomenal (i.e., spatio-temporal) entities."

Number. The following developments of the number concept are outlined in detail: (1) logistic [Frege and Russell-Whitehead], (2) empirico-postulational [Pasch] and (3) pure-postulational [Neder and Huntington]. These developments are compared carefully and the fundamental concepts involved in each of them are listed [pp. 38, 46, 49, 60]. Extensions of the number system are described, with indication given of both the algebraic and geometric "discovery" of each extension. The theory of aggregates and transfinite numbers is outlined and five of the classical paradoxes are discussed.

Logic. Detailed synopses are given of the following approaches to the foundations of mathematics: (I) Logistic [Russell-Whitehead] and nominalistic [Chwistek], (II) Formalistic [Hilbert] and elementalistic [Church] and (III) Intuitionistic [Heyting] and modal. These approaches are compared philosophically.

Special topics. A method is indicated for constructing undecidable propositions and the proof of Gödel's theorem is outlined. Mannoury's psycho-linguistic approach is analyzed in some detail. The concepts "symbol," "variable," "proof" and "definition" are discussed (with what should be considerable benefit to the writers of elementary mathematics texts). The significance and analysis of postulate systems are described. The author concludes with a treatment of the philosophical relation between mathematics and reality. *C. C. Torrance* (Annapolis, Md.).

*Cavaillès, Jean. **Transfini et continu.** Actualités Sci. Ind., no. 1020. Hermann et Cie., Paris, 1947. 24 pp.

This is a paper of expository type. Starting with the work of Dedekind and Cantor, the author discusses some of the fundamental concepts and difficulties of set theory, taking into account the recent work of Gödel on the continuum hypothesis. He also speaks of Gentzen's consistency proof of elementary number theory, and of the work on recursive functions by Herbrand, Church, and Kleene. Also touched upon are some problems connected with intuitionism and constructivism. *R. M. Martin* (Philadelphia, Pa.).

Carnap, Rudolf. On the application of inductive logic. Philos. and Phenomenol. Res. **8**, 133–148 (1947). (English. Spanish summary)

The subject of this paper is the application of inductive logic to actual or assumed knowledge situations. The first part of the paper is taken up with a discussion of certain semantical conditions of independence, simplicity and completeness that must be fulfilled by any language to which the author's system of inductive logic is to be applied. If we wish to apply the degree of confirmation $c(h, e)$ (of a hypothesis h with respect to the evidence e) to a given knowledge situation regarding h it is necessary that we take as e the total relevant evidence available to us (principle of total evidence). The author rejects certain objections to his theory of confirmation raised by N. Goodman [J. Philos. **43**, 383–385 (1946)] by pointing out violations of this principle of total evidence. The paper concluded with a general discussion of the notion of inductive projectibility, a property W being called inductively projectible if, for a fixed evidence, the probability that a nonobserved individual have the property W is a monotonically increasing function of the observed relative frequency of W. The author claims that in his system it is not necessary to restrict the inductive procedure to projectible properties. *O. Helmer.*

Goodman, Nelson. On infirmities of confirmation-theory. Philos. and Phenomenol. Res. **8**, 149–151 (1947).

This is a reply to Carnap's paper reviewed above, in which the author declares himself unconvinced that his earlier objections to Carnap's theory of induction had been adequately met. *O. Helmer* (Santa Monica, Calif.).

Carnap, Rudolf. Reply to Nelson Goodman. Philos. and Phenomenol. Res. **8**, 461–462 (1948).

Cf. the preceding review.

Finsler, P. Über die mathematische Wahrscheinlichkeit. Elemente der Math. **2**, 108–114 (1947).

After rejecting the Laplacean and frequency definitions of probability on account of their well-known difficulties, the author sets out to lay the axiomatic foundations of probability on a basis similar to the Laplacean but with changes which remove the difficulties of the latter without materially adding to its complication. After formulating as fundamental the principle that the axioms of probability can only pass from assumed probabilities of some events to those of other events related to the former, and can never say what the original probabilities must be (except in the trivial cases of certainty and impossibility), the author lays down the following axioms for his set of events: (1) the existence and nonnegativeness of the probability of any event; (2) additivity of probabilities of two disjoint events; (3) probability of certainty is unity; (4) if each of two equally probable events implies a third event, then the conditional probabilities of the two events (given the third) are still equal. On this basis the multiplicative and other laws of probability are derived.

The author seems to be insufficiently acquainted with the recent literature; otherwise he could scarcely have avoided giving references to works which contain essentially his axioms stated in perfectly precise form; nor would he have stated that all axiomatic treatments appearing before the present paper were essentially unsuccessful attempts to rehabilitate the Laplace definition. [Cf. Koopman, Ann. of Math. (2) **41**, 269–292 (1940); **42**, 169–187 (1941); Bull. Amer. Math. Soc. **46**, 763–774 (1940); these Rev. **1**, 245; **2**, 227, 106]. *B. O. Koopman* (New York, N. Y.).

ALGEBRA

Chater, N., and Chater, W. J. A chain rule for use with determinants and permutations. Math. Gaz. **31**, 279–287 (1947).

The rule, given without proof, may be stated as follows: in the expansion of the determinant having a_s^r in row r, column s, the sign of the term $a_{s_1}^1, a_{s_2}^2 \cdots a_{s_n}^n$ is + or −, according as the number of cycles in the expression of the permutation $(s_1 s_2 \cdots s_n)$ differs from n by an even or odd number. [Reviewer's note: if the permutation has c_k cycles of order k, then $c_1+2c_2+\cdots+nc_n=n$, and

$$n-(c_1+c_2+\cdots+c_n)=c_2+2c_3+\cdots+(n-1)c_n;$$

so the rule reduces to the usual one on the evenness or oddness of the permutation by noticing that a k-cycle is equivalent to $k-1$ transpositions.] *J. Riordan.*

Thijsen, W. P. On a determinant. Nieuw Tijdschr. Wiskunde **35**, 210–216 (1947). (Dutch)

The transformation $y_\nu=\sum\gamma_{\nu\mu}x_\mu$ $(\nu, \mu=1, \cdots, n)$ yields a transformation in the N-dimensional linear space of the homogeneous polynomials in $x_1, \cdots, x_n$ of degree s, where $N=(s+n-1)!/(n-1)!s!$. It is proved that its determinant equals $(\det \gamma_{\nu\mu})^t$, where $t=(s+n-1)!/n!(s-1)!$.

N. G. de Bruijn (Delft).

Tibiletti, Cesarina. Una condizione caratteristica per le equazioni di quarto grado risolubili elementarmente. Period. Mat. (4) **25**, 214–223 (1947).

Duschek, A. Matrizen, Vektoren und Tensoren. Österreich. Ing.-Arch. **1**, 371–382 (1947).

This is a polemic attack on what the author regards as the false use of the notions of matrix, vector and tensor in some current works on applied mathematics and also on the introduction of specialized notations by omitting the indices usual in tensor theory. The validity of this most recent in a long line of attacks suffers, in the opinion of the reviewer, from the failure to consider matrices other than those introduced to describe linear transformations of one set of variables into a covariant set and from a similar failure to note the effectiveness of the tensor notation and algebra in dealing with a number of variables, quite apart from the prior presence of a group of transformations. In his analysis of the conceptual inadequacy and computational limitations of the classical type of vector analysis, the author is on surer ground. *W. Givens* (Knoxville, Tenn.).

Dines, Lloyd L. On a theorem of von Neumann. Proc. Nat. Acad. Sci. U. S. A. **33**, 329–331 (1947).

The author derives from an earlier result on convex bodies [Bull. Amer. Math. Soc. **42**, 353–365 (1936)] a simple proof of the following theorem of von Neumann. Let a_{ij} and b_{ij} be two real m by n matrices of which the first is positive $(a_{ij}>0)$. Then there exists a unique λ and nonzero, non-negative vectors $x=(x_1, \cdots, x_m)$, $y=(y_1, \cdots, y_n)$ such that $\lambda\sum_{j=1}^m a_{ij}x_j\geqq\sum_{j=1}^n b_{ij}x_j, i=1, \cdots, n$ and $\lambda\sum_{i=1}^n a_{ij}y_i\leqq\sum_{i=1}^n b_{ij}y_i$, $j=1, \cdots, m$. *L. H. Loomis* (Cambridge, Mass.).

Siddiqi, M. Raziuddin. On the field of operators in the theory of quaternions. Proc. Nat. Inst. Sci. India **7**, 223–236 (1941).

The "field" of operators mentioned in the title consists of the forty-one obtained by combining "the operations of taking the adjoint [=conjugate], the inverse, the opposite [=negative], the versor [=normalized quaternion], the scalar part and the vector part of a quaternion." Tables of products are given. Methods and results are elementary.

W. Givens (Knoxville, Tenn.).

Landsberg, P. T. On generalised E-numbers. Proc. London Math. Soc. (2) **49**, 458–478 (1947).

This paper is concerned with the structure of the enveloping algebra of a set of linear transformations $S_1, \cdots, S_n$ where $S_jS_i=-S_iS_j$ for $i\neq j$, and $S_i^2=s_iI$ with s_i in the base field F of all complex numbers and I the identity transformation. Many known results are repeated and the only novelty of the paper is the study of the case where some of the defining constants s_i may be zero.

A. A. Albert (Chicago, Ill.).

Turnbull, H. W. Note on the simultaneous system of two quadratic quaternary forms. J. London Math. Soc. **22**, 147–152 (1947).

The author [Proc. London Math. Soc. (2) **18**, 69–94 (1919)] gave a list of 125 forms in terms of which every rational integral concomitant of two quaternary quadratics, for the projective group of transformations, can be expressed rationally and integrally. Later the reviewer showed that the number 125 could be reduced by 3 [same J. **4**, 182–183 (1929)]. In this paper the author shows that five more of these forms are reducible and gives the reductions. The methods used are those of symbolic invariant theory.

J. Williamson (Flushing, N. Y.).

Turnbull, H. W. Note on the simultaneous system of two quaternary quadratics. Addendum. J. London Math. Soc. **22** (1947), 163–165 (1948).

In this the author gives the reduction of another concomitant of two quaternary quadratics. [Cf. the preceding review.] *J. Williamson* (Flushing, N. Y.).

Lee, H. C. On the composition of quadratic forms. Proc. Nat. Acad. Sci. U. S. A. **33**, 379–381 (1947).

Let $g(u_1, \cdots, u_p)$ be a real nonsingular quadratic form of index t, $0<t<p$. A case of a Hurwitz problem is to find a real nonsingular quadratic form $f(x_1, \cdots, x_n)$ such that $g(u)\cdot f(x)=f(y)$, where $y_1, \cdots, y_n$ are bilinear in $x_1, \cdots, x_n$, and $u_1, \cdots, u_p$. Dubisch [Ann. of Math. (2) **47**, 510–527 (1946); these Rev. **8**, 62] gave necessary and sufficient conditions for this problem to be solvable in the real. A discussion of these conditions leads the author to the following result. Write $n=u\cdot 2^{4\alpha+\beta}$, u odd, $\beta=0, 1, 2, 3$. The Hurwitz problem is solvable in the real if and only if $p\leqq 8\alpha+2^\beta$ if $t\equiv 0 \bmod 4$; $p\leqq 8\alpha+2^\beta-3[\beta/3]$ if $t\equiv 1$; $p\leqq 8\alpha+2\beta$ if $t\equiv 2$; $p\leqq 8\alpha+\beta^2-2[\beta/3]$ if $t\equiv 3$. *G. Pall* (Chicago, Ill.).

Abstract Algebra

Nachbin, Leopoldo. Une propriété caractéristique des algèbres booléiennes. Portugaliae Math. **6**, 115–118 (1947).

It is known that in a distributive lattice every maximal ideal is prime, and that in a Boolean algebra every prime ideal is also maximal. It is here shown that if every prime ideal in a distributive lattice L is maximal, then L is a Boolean algebra, i.e., "complements" can be defined.

R. Arens (Los Angeles, Calif.).

Diamond, A. H., and McKinsey, J. C. C. Algebras and their subalgebras. Bull. Amer. Math. Soc. **53**, 959–962 (1947).

An algebra Γ is constructed which is not a Boolean algebra, but such that every subalgebra of Γ with two generators is a Boolean algebra. This shows that it is not possible to give a set of axioms for Boolean algebra where each axiom is a universal sentence with only two variables. Similar results hold for Abelian groups, rings, distributive lattices, modular lattices, Browerian algebras, and closure algebras. It is also shown (using the algebra Γ) that every set of axioms for the classical sentential calculus must contain at least one axiom using more than two variables.

B. Jónsson (Providence, R. I.).

Dieudonné, Jean. Sur les produits tensoriels. Ann. Sci. École Norm. Sup. (3) **64** (1947), 101–117 (1948).

Let E and F be modules over a commutative ring A. In the present paper properties of the tensor product $E \oplus F$ are studied. An element $x \in E$ is called free if $\lambda x = 0$ implies $\lambda = 0$. If $x \in E$ and $y \in F$ are free then it is proved that $\lambda(x \oplus y) \neq 0$ for every non-nilpotent $\lambda \in A$, but examples are given to show that $x \oplus y$ is not necessarily free even though x and y are. An integral domain is called a Dedekind ring if every ideal in it can be uniquely expressed as a product of prime ideals. Tensor products of spaces (especially ideals) over Dedekind rings are considered, with major emphasis on determining certain indices (order and height) of a tensor product $E \oplus F$ with respect to a particular prime ideal $\mathfrak{p}$ in terms of the indices of the factors. *R. M. Thrall.*

Grundy, P. M. On integrally dependent integral domains. Philos. Trans. Roy. Soc. London. Ser. A. **240**, 295–326 (1947).

Let $\mathfrak{S}$ be an integral domain integral over the ring $\mathfrak{R}$ and let the quotient field L of $\mathfrak{S}$ be a finite separable extension of the quotient field K of $\mathfrak{R}$, with $L:K = n$. The notions of different-ideal, discriminant-ideal, complementary module, conductor, etc., are developed without the classical chain condition assumptions; norm theory is developed as a contribution to the problem of finding sufficient conditions for $\mathfrak{S}$ to have an n-term $\mathfrak{R}$-basis. In the ramification theory, i.e., the theory which is concerned with the possibility that $(\mathfrak{S} \cdot \mathfrak{p})_{\mathfrak{q}} \neq \mathfrak{q}$ for a prime ideal $\mathfrak{q}$ of $\mathfrak{S}$ lying over a prime ideal $\mathfrak{p}$ of $\mathfrak{R}$, the notion of convergent and strongly convergent prime ideals replaces, and generalizes considerably, the algebro-geometric maximal chain condition. It is shown, for example, that when $\mathfrak{p}$ is strongly convergent, the different-ideal of $\mathfrak{S}$ over $\mathfrak{R}$ is not contained in $\mathfrak{q}$ if and only if $(\mathfrak{S} \cdot \mathfrak{p})_{\mathfrak{q}} = \mathfrak{q}$ and the residue field of $\mathfrak{q}$ is separable over that of $\mathfrak{p}$; a prime ideal $\mathfrak{p}$ of $\mathfrak{R}$ is said to be convergent if $\lim (\mathfrak{a}^r)_{\mathfrak{p}} = (0)$ for every $\mathfrak{R}$-ideal $\mathfrak{a} \subseteq \mathfrak{p}$ with finite $\mathfrak{R}$-basis, and strongly convergent if $\lim (\mathfrak{p}^r)_{\mathfrak{p}} = (0)$ for $r \to \infty$. By $\mathfrak{a}_{\mathfrak{p}}$ is meant $\mathfrak{R}_{\mathfrak{p}} \cdot \mathfrak{a} \cap \mathfrak{R}$. Again (as an example) the ramification rank of $\mathfrak{q}$ is defined for a convergent prime ideal $\mathfrak{p}$; for $\mathfrak{p}$ maximal and θ any element of $\mathfrak{S}$, the factorization mod $\mathfrak{p}$ of the characteristic polynomial $f(x)$ of θ is given in terms of the minimal polynomials $g_i(x)$ of θ mod $\mathfrak{q}_i$ and the ramification ranks ρ_i of $\mathfrak{q}_i$: in fact, $f(x) = \prod (g_i(x))^{\rho_i/k_i}(\mathfrak{p})$, where k_i is the degree of $g_i(x)$. An appendix giving the basic ideal theory of the paper is conveniently included.

A. Seidenberg (Berkeley, Calif.).

Samuel, Pierre. Une généralisation des polynomes de Hilbert. C. R. Acad. Sci. Paris **225**, 1111–1113 (1947).

Let A be a commutative ring satisfying the descending chain condition; if $\mathfrak{a}(n)$ denotes the set of forms of degree n belonging to a homogeneous ideal $\mathfrak{a}$ in the polynomial ring $A[x_1, \cdots, x_s]$, then $\mathfrak{a}(n)$ is an A-module of finite length $\phi_{\mathfrak{a}}(n)$. It is then proved that (1) $\phi_{\mathfrak{a}}(n)$ is a polynomial in n for large n; (2) if $\mathfrak{b} = (x_1, \cdots, x_s)$, then $\phi_{\mathfrak{a}}(n) = \phi_{\mathfrak{a}:\mathfrak{b}}(n)$ for large n.

If $\mathfrak{O}$ is a Noetherian ring and $\mathfrak{q}$ a primary ideal with associated prime $\mathfrak{p}$, then it is proved that for n sufficiently large: (3) the length $P_{\mathfrak{q}}(n)$ of the symbolic power $\mathfrak{q}^{(n)}$ is a polynomial in n; (4) if $\mathfrak{p}$ is not an associated prime ideal of the intersection of all the primaries belonging to $\mathfrak{p}$, then $\mathfrak{q}^{(n+1)} : \mathfrak{q} = \mathfrak{q}^{(n)}$. By the standard quotient ring formation one may reduce the proofs of (3) and (4) to the special case where $\mathfrak{O}$ is a local ring and $\mathfrak{p}$ is its maximal ideal. The proof for this case is deduced from (1) and (2) by noting that $A = \mathfrak{O}/\mathfrak{q}$ satisfies the descending chain condition and by using the fact that the direct sum of the A-modules $\mathfrak{q}^n/\mathfrak{q}^{n+1}$ $(n = 0, 1, \cdots)$ can be made into a ring which is a homomorphic image of a polynomial ring over A.

I. S. Cohen (Cambridge, Mass.).

Samuel, Pierre. Sur les anneaux locaux. C. R. Acad. Sci. Paris **225**, 1244–1245 (1947).

Let $\mathfrak{O}$ be a local ring, $\mathfrak{m}$ its maximal ideal, $\mathfrak{q}$ any ideal primary for $\mathfrak{m}$. Then the degree d of $P_{\mathfrak{q}}(n)$ [see the preceding review] is the same for all $\mathfrak{q}$; d is called the dimension of $\mathfrak{O}$. It is shown that no set of less than d elements generates an ideal primary for $\mathfrak{m}$ and that there exist sets of d elements (called systems of parameters) which do generate such a primary; this shows the equivalence of this definition of dimension with that of Chevalley [Ann. of Math. (2) **44**, 690–708 (1943); these Rev. **5**, 171]. The highest term of $P_{\mathfrak{q}}(n)$ has the form $e(\mathfrak{q})n^d/d!$, where $e(\mathfrak{q})$ is a positive integer; it is called the multiplicity of $\mathfrak{q}$. This generalizes the definition of Chevalley [Trans. Amer. Math. Soc. **57**, 1–85 (1945); these Rev. **7**, 26]. If $\mathfrak{O}$ satisfies the condition that $\dim \mathfrak{O}/\mathfrak{p} + \dim \mathfrak{O}_{\mathfrak{p}} = \dim \mathfrak{O}$ for every prime ideal $\mathfrak{p}$, and if $\mathfrak{O}/\mathfrak{m}$ is infinite, then for every $\mathfrak{q}$ there exists a $\mathfrak{q}' \subset \mathfrak{q}$ generated by a system of parameters and such that $e(\mathfrak{q}') = e(\mathfrak{q})$. By generalizing Chevalley's concept of a geometric local ring [loc. cit.], further extensions of his results are obtained.

I. S. Cohen (Cambridge, Mass.).

Loonstra, F. Représentation d'éléments $\mathfrak{p}$-adiques et $\mathfrak{q}$-adiques. Nederl. Akad. Wetensch., Proc. **50**, 1117–1125 = Indagationes Math. **9**, 531–539 (1947).

Let O be an integral domain in which every ideal is a unique product of maximal prime ideals; let $K_{\mathfrak{p}}$ be the completion of the quotient field of O with respect to the valuation defined by the powers of a prime ideal $\mathfrak{p}$ in O. It is shown how to represent the elements of K by nested sequences of equivalence classes and by infinite series. The results and proofs are not different from the case where O is the ring of ordinary integers [cf. B. L. van der Waerden, Moderne Algebra, 2d ed., Springer, Berlin, 1937, v. 1, pp. 251–254]. The unnecessary restriction that $\mathfrak{p}$ be a principal ideal is sometimes made. Let $O_{\mathfrak{q}}$ be the completion of O with respect to the pseudo-valuation defined by the powers of an arbitrary ideal $\mathfrak{q}$. It is shown that $O_{\mathfrak{q}}$ is the direct sum of the rings $O_{\mathfrak{p}_i}$, where the $\mathfrak{p}_i$ are the distinct prime factors of $\mathfrak{q}$. This result is essentially implied by Chevalley's work on semi-local rings [Ann. of Math. (2) **44**, 690–708 (1943), especially proposition 8; these Rev. **5**, 171].

I. S. Cohen (Cambridge, Mass.).

Cartan, Henri. Théorie de Galois pour les corps non commutatifs. Ann. Sci. École Norm. Sup. (3) **64**, 59–77 (1947).

This paper contains the proofs of the results stated by the author in two previous papers [C. R. Acad. Sci. Paris **224**, 87–89, 249–251 (1947); these Rev. **9**, 5]. The main tool used in the proofs is a generalization due to N. Bourbaki to noncommutative fields K of a theorem of Jacobson. Let $E(K)$ be the ring of endomorphisms of the additive group of K, with its structure of right vector space over K. A subring of $E(K)$ is called a vector subring if (1) it contains the identity; (2) it is a subspace of the vector space $E(K)$. The theorem of Jacobson-Bourbaki establishes a one-to-one correspondence between the subfields L of K over which K

is of finite left dimension and the vector subrings of $E(K)$ which are finite dimensional vector spaces over K. On the other hand, use is made of a theorem of Artin's type which gives the condition under which a system of automorphisms of a noncommutative field can be linearly dependent [lemma 1, p. 68]. *C. Chevalley* (Princeton, N. J.).

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions galoisiennes des corps de nombres algébriques: anneau principal; lois d'unicité, d'ordination, d'existence (forme provisoire), d'isomorphisme et de décomposition; loi de monodromie. C. R. Acad. Sci. Paris **225**, 1113–1115 (1947).

Using the notation and terminology of his previous notes [same C. R. **225**, 785–787, 973–975 (1947); these Rev. **9**, 223], the author considers the principal ring R_k which consists (up to equivalence) of the representations r_Γ associated with the representations Γ of the Galois groups $G_{K/k}$ of all normal extensions K of k. The ring $\mathfrak{T}_{K/k}$ of representations r_Γ of $U_{K/k}$ associated with the representations Γ of $G_{K/k}$ is a positively complete subring of R_k and is called the Takagi ring of K/k. If K, K' are two normal extensions of k then $\mathfrak{T}_{K/k} \subset \mathfrak{T}_{K'/k}$ if and only if $K \subset K'$, and $\mathfrak{T}_{K/k} = \mathfrak{T}_{K'/k}$ if and only if $K = K'$ (laws of ordering and uniqueness). Moreover, if $\mathfrak{T}$ is any positively complete subring of R_k there exists a normal extension K of k such that $\mathfrak{T}$ is the Takagi ring of K/k (law of existence). A partial law of isomorphism and the law of decomposition are also given.

Two subextensions L and L' of an extension L^* of an algebraic or $\mathfrak{p}$-adic field l are said to be lateral if both LL'/L and LL'/L' are unramified. If $\mathfrak{p}$ is a prime ideal (finite or infinite) of k and $\mathfrak{P}'$ is a prime divisor of $\mathfrak{p}$ in an extension field K' of k then the $\mathfrak{P}$-adic extension $K'_{\mathfrak{P}'}/k_{\mathfrak{p}}$ is normal if K'/k is normal and in this case is denoted by $(K'/k)_{\mathfrak{p}}$ since it depends only on $\mathfrak{p}$ and K'. Denoting by D_K, $D_{K/k}$, and $N(—)$ the absolute and relative discriminants, and absolute norm, respectively, the following monodromic law is proved. Let K, K' be two extensions of k of which K/k is normal of degree n. If, for every prime divisor $\mathfrak{P}'$ of $D_{K/k}$ in K', $K'_{\mathfrak{P}'}/k_{\mathfrak{p}}$ is lateral to $(K/k)_{\mathfrak{p}}$, and if, for every prime ideal $\mathfrak{P}'$ of K' such that $N(\mathfrak{P}') \leqq |D_{K'}|^{n/2}$, $K'_{\mathfrak{P}'}/k_{\mathfrak{p}}$ is an extension of $(K/k)_{\mathfrak{p}}$, then $K' \supset K$. *D. C. Murdoch* (Vancouver, B. C.).

THEORY OF GROUPS

Vilenkin, N. Ya. On the theory of general topological groups. Doklady Akad. Nauk SSSR (N.S.) **58**, 1573–1575 (1947). (Russian)

This note begins with a proof of the following lemma. Let G be a discrete Abelian group, H a subgroup. Suppose, further, that T is an infinitely divisible group (i.e., $nT = T$, for every integer n) such that there exists an isomorphic mapping f of H into T. Then there exists a mapping g of G into $\bar{G} = T + G_1$ (G_1 is isomorphic to the factor group G/H) such that $g(G) \cap T = f(H)$. The isomorphism is constructed by a transfinite induction based on the ascending chain of groups $H = H_0 \subset H_1 \subset \cdots$, such that $G = \cup_\alpha H_\alpha$ and the factor group associated with two consecutive ones of these groups is finite or infinite cyclic. The first step in the construction of the mapping g as a "limit" of mappings: $g_0, g_1, \cdots$, is the obvious one: $g_0(x) = f(x)$, if $x \in H_0$. The lemma leads to the following theorem. Let G be a general topological Abelian group. Then there exists a Hausdorff topological group G' and subgroup H' (not necessarily closed) such that $G = G'/H'$. For definitions of terms, there is a reference to S. Eilenberg and S. MacLane [Ann. of Math. (2) **43**, 757–781 (1942), in particular, p. 761; these Rev. **4**, 88]. *L. Zippin.*

Braconnier, Jean. Cascades de groupes topologiques. C. R. Acad. Sci. Paris **225**, 1250–1251 (1947).

A continuation of a previous note [same C. R. **224**, 370–372 (1947); these Rev. **8**, 438]. A cascade is a family of groups $\{G_\alpha\}$ satisfying conditions L_1 and L_{11} of that note. These conditions involve a filter relation "$\leqq$" on the subscripts α and for each $\alpha \leqq \beta$ a representation $f_{\alpha\beta}$ of an open subgroup $H_{\alpha\beta}$ of G_α into G_β. It is also assumed that the $f_{\alpha\beta}$ compose reasonably. With this setup a certain quotient group G of two subgroups of $\prod G_\alpha$ is defined and called the limit of the cascade. In this paper it is first proved that, if $\{L_\lambda\}$ is another cascade with limit group L, and if an isomorphism of the filter of α's into a fundamental subset of the filter of λ's is given, together with a representation of each G_α into the corresponding L_λ, and satisfying some obvious conditions, then this defines a unique representation of G into L. Next it is shown that, under certain conditions, the limit of a cascade of quotients is the quotient of the limits of the cascades. Finally, it is shown that if a locally compact group G admits a one-to-one representation into a limit of a cascade $\{L_\lambda\}$ then G is itself a limit of a cascade in a significant way. *W. Ambrose* (Cambridge, Mass.).

Cartan, Henri, et Godement, Roger. Théorie de la dualité et analyse harmonique dans les groupes abéliens localement compacts. Ann. Sci. École Norm. Sup. (3) **64**, 79–99 (1947).

An elegant short treatment of the Fourier theory on locally compact Abelian groups including its use to prove, without any structure theorems, the Pontrjagin duality theorem. First, following Gelfand and Raikov [Rec. Math. [Mat. Sbornik] N.S. **13**(**55**), 301–316 (1943); C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 199–201 (1944); these Rev. **6**, 147] the Krein-Milman theorem [Studia Math. **9**, 133–138 (1940); these Rev. **3**, 90], which says that certain convex sets are generated by their extreme points, is applied to a certain space P of positive definite functions on the group G. The extreme points in this case are precisely the continuous characters of G so this yields the existence of sufficiently many characters. Then a new application is made of the Krein-Milman theorem to obtain a simple proof of the Raikov-Herglotz-Bochner theorem on the representation of positive definite functions; this theorem follows from the Krein-Milman theorem because the continuous characters admit the desired representation and the set of functions admitting such a representation is convex.

The topology of the character group G^* is defined as that induced by the weak topology on L^*; it is easily proved to be the usual topology and to be locally compact. The Fourier transform f^* of an integrable function f on G and the Fourier transform μ of a bounded complex measure μ^* on G^* are defined through the usual formulas; f^* is a function on G^* and μ a function on G. Through Stone's theorem it is shown that μ uniquely determines μ^*. Letting V be the class of complex linear combinations of functions in P this, with the Raikov-Herglotz-Bochner theorem, shows that every f in V is the Fourier transform of a unique measure μ^* on G^*. Further argument shows, if f is also integrable, that this complex measure is the one obtained by integrating

f^* (the Fourier transform of f) with respect to the Haar measure of G^*: this is the Fourier inversion formula. With this basis the Plancherel and Pontrjagin duality theorems follow readily. Several other results in harmonic analysis are proved.

In papers unavailable to these authors Raikov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 589–591 (1941); Trav. Inst. Math. Stekloff **14** (1945); these Rev. **2**, 317; **8**, 133] had earlier used harmonic analysis to obtain the Pontrjagin duality theorem. His methods were different although he states (in the second reference) the possibility of a development based on the Gelfand-Raikov work which is used here. *W. Ambrose* (Cambridge, Mass.).

Godement, Roger. Théorèmes taubériens et théorie spectrale. Ann. Sci. École Norm. Sup. (3) **64** (1947), 119–138 (1948).

The ideal theory of the L_1 group algebra of a locally compact Abelian group (i.e., the collection of complex-valued functions on the group which are integrable relative to Haar measure, regarded as an algebra with convolution as multiplication) is investigated and results are obtained which in the case of the additive group of the reals coincide with the first of Wiener's general Tauberian theorems and with a theorem related to an extension of Wiener's theorem due to Beurling. The author's main results about L_1 are: (1) if C^* is a compact subset of the character group G^* of G, and if f is a function in L_1 whose Fourier transform F does not vanish on C^*, then there exists an element $h \in L_1$ whose Fourier transform F satisfies $H(x^*)F(x^*)=1$ for $x^* \in C^*$; (2) if I is a closed ideal in L_1 such that for each $x^* \in G^*$ there is an element of I whose Fourier transform does not vanish at x^*, then $I=L_1$ (essentially Wiener's theorem in case G is the additive group of the reals). [These results are contained in somewhat more general form in a recent paper of the reviewer, Trans. Amer. Math. Soc. **61**, 69–105 (1947); summarized in Proc. Nat. Acad. Sci. U. S. A. **27**, 348–352 (1941); these Rev. **8**, 438; **3**, 36, which was not known to the author until after the submission of his paper for publication.]

These results permit conclusions to be drawn about group invariant linear subspaces F of L_∞ (the space of essentially bounded measurable functions on G) which are closed in the weak topology of L_∞ as the dual of L_1, by virtue of the fact that the set of elements of L_1 which annihilate F is a closed ideal in L_1 whose annihilator is F. It is concluded, using results about weak topologies, both in Banach spaces generally and in L_∞ in particular, that if φ is a bounded continuous function on G then there exists a directed set of linear combinations of translations of φ, each of which is bounded by unity, which converges uniformly on every compact set to a character of G. This is similar to Beurling's theorem [Acta Math. **77**, 127–136 (1945); these Rev. **7**, 61] except that Beurling, who restricts himself to a uniformly continuous φ on the reals, obtains convergence of the bounds to unity, an aspect which is significant for some applications. Continuing along these lines the author shows that, defining the spectrum of an element $f \in L_\infty$ as the set of characters contained in the smallest group-invariant weakly closed linear subspace of L_∞ which contains f (this is similar to a definition of Beurling's), the following is true: (1) if the spectrum of f is compact then f is almost everywhere equal to a uniformly continuous function; (2) the spectrum of the sum of two functions is contained in the union of the spectra of the functions and equals the union in case the spectra are disjoint. *I. E. Segal* (Chicago, Ill.).

Godement, Roger. Les fonctions de type positif et la théorie des groupes. Trans. Amer. Math. Soc. **63**, 1–84 (1948).

Analogues to various known results concerning positive definite functions on the reals are formulated and proved for the case of positive definite functions on an arbitrary locally compact group G. The set P of all continuous positive definite functions on G which are bounded by 1 in absolute value constitutes a convex set whose extreme points are called "elementary." It is proved that every $f \in P$ is a limit, uniformly on every compact set, of a (directed set of) linear combinations with positive coefficients of elementary functions. From this it follows that every continuous function is a limit uniformly on every compact set of linear combinations of elementary functions. (In case G is compact, this is essentially the Peter-Weyl theorem.) For every $f \in P$ there is a regular measure μ on the closure (in the weak topology on the conjugate space of the space $L_1(G)$ of complex-valued functions on G integrable relative to Haar measure) E of the totality of elementary functions such that for any $\varphi \in L_1(G)$, $(f, \varphi)=\int_E (e, \varphi)d\mu(e)$, where for any function ψ on G, $(f, \psi)=\int_G f(x)\psi(x)(\rho(x))^{\frac{1}{2}}dx$, dx being the element of Haar measure, and ρ being defined by the equation $dx^{-1}=\rho(x)dx$. (In case G is Abelian, this is essentially the Herglotz-Bochner-Weil theorem; see Weil, L'integration dans les groupes topologiques, Actualités Sci. Ind., no. 869, Hermann, Paris, 1940, p. 122; these Rev. **3**, 198. The measure μ will in general not be unique, and for this and other reasons pointed out by the author the result cannot be regarded as an altogether satisfactory analogue of that theorem.)

For any $f \in P$ the author defines a kind of mean $M(f)$ similar to the mean of an almost periodic function, and every $f \in P$ has the form $f=g+h$, where g is almost periodic and $M(|h|^2)=0$ and g and h are in P. If $f \in P$, there exists a $g \in L_2(G)$ (space of complex-valued square integrable functions on G) such that $f=g*g^\sim$, where $g^\sim(x)=\bar{g}(x^{-1})$. If f and g are both in $P \cap L_2(G)$, then $f*g \in P$. Every element of $P \cap L_2(G)$ is the limit in $L_2(G)$ of a linear combination with positive coefficients of elements f of $P \cap L_2(G)$ with the property that $f*f^\sim=f$. A topology (not in general Hausdorff) is defined for the aggregate of unitary-equivalence classes of continuous irreducible unitary representations of G; it is derived from the weak topology in the conjugate space of $L_1(G)$ applied to the elementary functions and the mutual correspondence between such representations and elementary functions.

Many of the methods and some of the theorems of the author (notably the first theorem quoted) represent refinements of the work of Gelfand and Raikov [Rec. Math. [Mat. Sbornik] N.S. **13(55)**, 301–316 (1943); these Rev. **6**, 147], who first established the existence of elementary functions, and showed the existence of a natural mutual correspondence between elements of P and continuous unitary representations of G, in which elementary functions correspond to irreducible representations. Weak topologies on functions and measure on locally compact spaces are utilized frequently by the author. Actually many of the author's results were originally obtained independently of Gelfand and Raikov and the paper includes, in addition to the results described above, a complete exposition of the work of Gelfand and Raikov and of related work.

I. E. Segal (Chicago, Ill.).

Raĭkov, D. A. On various types of convergence of positive definite functions. Doklady Akad. Nauk SSSR (N.S.) 58, 1279–1282 (1947). (Russian)

The function φ defined on a locally compact group G is called positive definite if, for arbitrary $g_i \in G$ and complex ξ_i, $\sum_{i=1}^{n}\sum_{l=1}^{n}\varphi(g_i g_l^{-1})\xi_i\bar{\xi}_l \geqq 0$. We distinguish several modes of convergence of sequences of such functions: (a) pointwise convergence, (b) uniform convergence in every compact set, (c) uniform convergence in the whole group. We denote by $L_1(G)$ the space of functions x integrable with respect to the left invariant Haar measure. All linear functionals in $L_1(G)$ are of the form $\Phi(x)=\int\varphi(g)x(g)dg$. This leads to three more modes of convergence of φ: (α) weak; (β) weak convergence together with convergence of the norm, $\lim \varphi_n(e)=\varphi(e)$;

$$(\gamma) \qquad \lim_{m,n}\sup_{|x|\leq 1}\left|\int[\varphi_n(g)-\varphi_m(g)]x(g)dg\right|.$$

The main theorems of this paper are the following: (c) is equivalent to (γ); (a) implies (α); if the limit function is continuous, then (a) implies (β); (b) is equivalent to (β); if a sequence of positive definite functions on a locally compact group is pointwise convergent, then it converges uniformly on every compact subset to a positive definite function. *František Wolf* (Berkeley, Calif.).

Mautner, F. I. The completeness of the irreducible unitary representations of a locally compact group. Proc. Nat. Acad. Sci. U. S. A. 34, 52–54 (1948).

J. von Neumann has introduced [as yet unpublished] the notion of a direct integral of Hilbert spaces H_t, with respect to a weight function $s(t)$, the dimension $k(t)$ of H_t being s-measurable: this is the Hilbert space $H=\int H_t$ consisting of all complex s-measurable functions $x_n(t)$ $(-\infty<t<+\infty, n=1,\cdots,k(t))$, with $\int\sum_n|x_n(t)|^2dt<\infty$. Every x in H is then a function $x(t)$ with $x(t)\in H_t$. A bounded operator A in H is said to decompose if $(Ax)(t)=A(t)x(t)$ with a bounded operator $A(t)$ in H_t for every t. The author announces the following theorems. (I) If F is a self-adjoint family of bounded operators on a Hilbert space H, then H can be written as a direct integral of spaces H_t such that every operator $A\in F$ decomposes, and such that s-almost all H_t are irreducible under the induced family $F(t)=\{A(t)\}$ of operators. (II) If U is a unitary representation of a group G on a separable Hilbert space H, then U is a generalized direct sum of irreducible unitary representations; if G is locally compact and U strongly continuous then s-almost all components $U(t)$ are strongly continuous. These results lead to a generalized Parseval relation for separable unitary representations, in particular, for the regular representation of a locally compact separable group, and to an expression of measurable positive definite functions as "integrals" of elementary (in the sense of Gelfand and Raikov) positive definite functions. *H. Samelson* (Ann Arbor, Mich.).

Gel'fand, I. M., and Naĭmark, M. A. Unitary representations of semisimple Lie groups. I. Mat. Sbornik N.S. 21(63), 405–434 (1947). (Russian)

Chapitre I. Construction explicite des mesures invariantes gauche et droite dans les groupes complexes unimodulaires G. § 1. Soit K le sous-groupe des matrices surdiagonales, H celui des matrices sousdiagonales, Z le sous-groupe de K, où les éléments diagonaux sont 1, Z le sous-groupe de H, où les éléments diagonaux sont 1, D le sous-groupe de G des matrices diagonales. Sortant du fait qu'une multiplication gauche transforme linéairement les paramètres de la matrice variable k, on calcule la mesure gauche en K,

$$d\mu_l(k)=|k_{22}|^2|k_{44}|^4\cdots|k_{nn}|^{2n-4}dudv,$$

$(u=\Re(k), v=\Im(k))$, et la mesure droite en K,

$$d\mu_r(k)=|k_{22}|^{-2}|k_{33}|^{-6}\cdots|k_{nn}|^{-2n}dudv.$$

Si f est sommable sur K, on a

$$\int f(kk_0)d\mu_l(k)=\beta(k_0^{-1})\int f(k)d\mu_l(k),$$

$$\int f(k_0k)d\mu_r(k)=\beta(k_0)\int f(k)d\mu_r(k),$$

où

$$(*) \qquad \beta(k)=d\mu_l(k)/d\mu_r(k)=|k_{22}|^4|k_{33}|^6\cdots|k_{nn}|^{4n-4}.$$

Pour H, il y a des formules analogues. En Z et Z les mesures invariantes sont $d\mu=dudv$ et pour D

$$d\mu=|\delta_2|^{-2}\cdots|\delta_n|^{-2}dudv,$$

où $\delta_1,\cdots,\delta_n$ sont les éléments diagonaux.

§ 2. Chaque élément de K peut être représenté d'une seule manière comme produit d'un élément δ de D et d'un élément de Z, où δ ne dépend pas de l'ordre du produit. Si l'on veut représenter un élément de G comme produit d'un élément de Z et d'un élément de H, il y a des éléments exceptionnels, dont l'un des mineurs

$$g_m=\begin{vmatrix} g_{mm} & \cdots & g_{mn} \\ \cdots & \cdots & \\ g_{nm} & \cdots & g_{nn}\end{vmatrix}$$

s'annule. Chaque élément k de K peut être représenté d'une seule manière comme $\zeta^{-1}\delta\zeta$, $\zeta\in Z$, $\delta\in D$, où les éléments diagonaux de k et δ sont les mêmes. Enfin on trouve pour les éléments g de G à racines différentes λ une représentation $z^{-1}kz$ $(k_{pp}=\lambda_p)$, si certains déterminants ne s'annulent pas.

§ 3. En se servant du théorème de Fubini on réduit l'intégration en K à celles en D et Z:

$$(**) \qquad \int_K f(k)d\mu_l(k)=\int_D d\mu(\delta)\int_Z f(\delta\zeta)d\mu(\zeta)$$

$$=\int_Z d\mu(\zeta)\int_D f(\delta\zeta)d\mu(\delta),$$

et des formules analogues pour $d\mu_r(k)$ et pour H, D, Z. Une analyse analogue donne

$$\int_G f(g)d\mu(g)=\int_H d\mu_r(h)\int_Z f(\zeta h)d\mu(\zeta)=\int_Z d\zeta\int_H f(\zeta h)d\mu_r(h)$$

et la formule correspondante pour G, K, Z et $d\mu_l(k)$. La combinaison de ces formules et celles du § 1 est:

$$\int_Z d\mu(z)\int_K f(z^{-1}kZ)\varphi(k)d\mu_l(k)=\int_G f(g)\frac{\sum\varphi(k_g)\beta^{\frac{1}{2}}(k_g)}{\prod_{\lambda<\varepsilon}|\lambda_g^{(\varepsilon)}-\lambda_g^{(\varepsilon)}|^2}d\mu(g),$$

où β est défini par (*) et k_g par $z^{-1}k_gz=g$ et où les λ sont les racines de g.

§ 4. Soit Z' la multiplicité des classes gauches de K en G. En Z' il y a une mesure invariante conforme à celle existante en Z, les éléments de Z' non représentés par des éléments de Z remplissant un ensemble de mesure 0. Théorème analogue pour H et H' (=système des classes droites de G sur Z).

Chapitre II. La série fondamentale des représentations irréductibles de G: $\mathbf{H}_{H'}$=espace unitaire des fonctions f définies sur H'; U_g=l'opérateur unitaire transformant $f(h')$ en $f(h'g)$; $\mathbf{H}_Z$=espace unitaire des fonctions f définies sur

Z; X=groupe des caractères $\chi(\delta)$ de D. Appliquant la décomposition $h'=\delta z$ (possible presque partout) à la formule (**) on est conduit à

$$\|f\|^2=\int_Z d\mu(z)\int_D|\beta^{\frac{1}{2}}(\delta)f(\delta z)|^2d\delta$$

pour les f des $\mathbf{H}_{H'}$. Donc $\varphi_z(\delta)=\beta^{\frac{1}{2}}(\delta)f(\delta z)$ est de la classe L_2 pour presque tous les $z\in Z$. Pour

$$f_\chi(z)=\int_D\varphi_z(\delta)\overline{\chi(\delta)}d\delta,$$

une extension du théorème de Plancherel donne

$$\int_X|f_\chi(z)|^2d\mu(\chi)=\int_D|\beta^{\frac{1}{2}}(\delta)f(\delta z)|^2d\delta=\int_X d\mu(\chi)\int_Z|f_\chi(z)|^2d\mu(z).$$

Cela conduit à une décomposition de $\mathbf{H}_{H'}$ et de la représentation U_g d'après les χ. Les $U_{\chi,g}$ ont la forme

$$U_{\chi,g}(f(z))=\alpha_\chi(zg)f(zg),$$

où l'on a posé $\alpha_\chi(g)=\beta^{-\frac{1}{2}}(g)\chi(g)$, $zg=z^1$ étant défini par la décomposition $g^1=zg=kz^1$, tandis que $\chi(g)$ est posé égal à $\chi(\delta)$ en vertu de $g=\zeta\delta_z$. Pour $U_{\chi,g}$ on trouve l'expression plus détaillée

$$U_{\chi,g}f(z_{pq})=|g_z{}^1|^{m_1+i\rho_1-2}|g_z{}^1|^{m_2-m_1+i(\rho_2-\rho_1)-2}\cdots$$
$$\times|g_n{}^1|^{m_{n-1}-m_{n-2}+i(\rho_{n-1}-\rho_{n-2})-2}g_z{}^{1-m_1}g_z{}^{1-(m_2-m_1)}\cdots$$
$$\times g_n{}^{1-(m_{n-1}-m_{n-2})}f(z^1_{pq}),$$

où les ρ et les entiers m dépendent du caractère χ respectif. Pour le cas de deux dimensions, cette formule devient

$$U_{\chi,g}f(z)=|g_{12}z+g_{22}|^{m+i\rho-2}(g_{12}z-g_{21})^{-m}f\left(\frac{g_{11}z+g_{21}}{g_{12}z+g_{22}}\right),$$

où $z=z_{21}$ en $\left(\begin{smallmatrix}1&0\\z_{11}&1\end{smallmatrix}\right)$. L'irréducibilité de ces représentations est démontrée par induction.

Dans l'introduction les auteurs mentionnent que les représentations irréductibles continues dans la série fondamentale doivent être complétées par une série de représentations unitaires qu'ils appellent supplémentaire et dégénérée. La totalité de ce système plus étendu sera démontrée dans la deuxième partie de ce mémoire. *H. Freudenthal.*

Gel'fand, I. M., and Naĭmark, M. A. Supplementary and degenerate series of representations of the complex unimodular group. Doklady Akad. Nauk SSSR (N.S.) **58**, 1577–1580 (1947). (Russian)

Les auteurs généralisent d'abord la notion de série fondamentale de représentations unitaires du groupe unimodulaire. En conservant les notations du mémoire revu ci-dessus, on définit alors les sous-groupes K comme il suit. Soient $n_1,\cdots,n_r$ des nombres naturels, $n_1+\cdots+n_r=n$ (si c'est le groupe unimodulaire de n variables qu'il faut représenter). Alors les g_{pq} ne sont pas les éléments du matrice $\|g_{pq}\|$, mais eux-mêmes des matrices de n_p lignes et n_q colonnes, et K est alors le groupe des matrices pour lesquelles la matrice $K_{pq}=0$ pour $p>q$ et $\prod\det(k_{pq})=1$. A cette généralisation près, toutes les définitions et notations sont des répétitions verbales de celles du mémoire cité. Les séries de représentations nouvelles sont appelées dégénérées; elles consistent toujours de représentations irréductibles. A part de celles-ci on trouve des séries supplémentaires. Soient $n_1=n_2=\cdots=n_{2r}=1$ ($r\geqq1$). Alors Z' est l'ensemble des éléments z' de Z, qui ont $z'_{2j,2j-1}\neq0$ pour $0\leqq j\leqq r$ et $z'_{pq}=0$ ailleurs; Σ est l'ensemble des paires $\{z,z'z\}$. Le groupe transforme Σ d'après la loi $\{z,z'z\}g=\{zg,(z'z)g\}$. Alors des définitions tout à fait analogues à celles du mémoire précédent conduisent à des représentations irréductibles constituant la série supplémentaire (non dégénérée pour $r=n$, dégénérée dans les autres cas). La note présente ne contient pas de démonstrations. *H. Freudenthal.*

Courtois, Jacques. Représentations linéaires du groupe affine. C. R. Acad. Sci. Paris **225**, 850–852 (1947).

Le rapporteur croit avoir compris ce qui suit: (I) soit G le groupe linéaire général de rang N (réel ou complexe); on en obtient une représentation unitaire irréductible (ce que l'auteur souligne!) en associant à la matrice A de G le nombre complexe $|\det(A)|^{i\sigma}$; (II) dans le cas du groupe réel, soit, dans l'espace vectoriel de dimension N, Ω la sphère $\|x\|=1$; soit H l'espace de Hilbert formé des fonctions numériques $f(x)$ définies sur Ω, telles que $\sum|f(x)|^2<+\infty$, $x\in\Omega$, avec le produit scalaire habituel; alors, on obtient une représentation unitaire de G en associant, à $A\in G$, l'opérateur $f(x)\to\|A^{-1}x\|^{i\sigma}f(A^{-1}x/\|A^{-1}x\|)$, où σ est un nombre réel définissant la représentation. Ces résultats sont exposés au moyen d'une terminologie et dans un style dont il est impossible de donner ici la plus faible idée; les notions introduites par l'auteur ("radieurs," "coradieurs," "expanseurs," etc.) sont, semble-t-il, totalement superflues (sinon pour préserver sa note de toute indiscrétion) et peuvent être exposées dans un langage relativement civilisé. Toutefois, il serait prématuré et imprudent d'attribuer à ce commentaire une valeur absolue, en raison de l'incertitude qui pèse sur les intentions de l'auteur. *R. Godement.*

Courtois, Jacques. Représentations linéaires du groupe affine complexe. C. R. Acad. Sci. Paris **225**, 1247–1249 (1947).

Le principal résultat de cette note semble être le suivant: soient G le groupe linéaire général (complexe) de rang N, G' le groupe unimodulaire complexe de même rang, g le groupe multiplicatif des nombres complexes non nuls; alors on obtient des représentations unitaires irréductibles de G en composant une représentation unitaire irréductible de G' avec un caractère de g. Outre que ce résultat ne soit pas d'une profondeur et d'une originalité considérables, il est inquiétant de constater que l'auteur n'est pas certain d'obtenir par ce procédé un système complet de représentations de G; sans doute cette remarque de l'auteur doit-elle être interprétée dans un sens humoristique.

R. Godement (Nancy).

de Kerékjártó, B. Sur les groupes compacts de transformations topologiques de surfaces en elles-mêmes. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 9–32 (1941). (Hungarian. French summary)

L'auteur démontre que les seules surfaces à connexion finie qui admettent des groupes compacts et infinis sont les suivantes: la sphère, la surface circulaire, la couronne circulaire, le tore, le plan projectif, la bande de Möbius et l'anneau non-orientable. Ce sont les mêmes types topologiques de surfaces qui admettent des groupes continus et transitifs.

Ensuite l'auteur détermine les groupes compacts infinis correspondant aux 7 types de surfaces énumérés ci-dessus. Les groupes compacts de la sphère ont été déterminés dans un mémoire précédent. La détermination des groupes compacts de la surface circulaire, de la couronne circulaire, du plan projectif et de la bande de Möbius peut être ramenée aux groupes compacts de la sphère; le cas de l'anneau non-

orientable à celui du tore. Il ne reste qu'à énumérer les groupes compacts du tore.

Les groupes finis du tore ont été énumérés par Brouwer. Concernant les groupes compacts et infinis du tore, ce mémoire contient leur énumération effective qui conduit au résultat général suivant. Tout groupe compact et infini G de transformations topologiques du tore en lui-même peut être exprimé comme un groupe de transformations linéaires de cordonnées bicirculaires convenablement choisies. Le groupe G peut être obtenu soit à partir du groupe des translations du tore en y ajoutant une ou deux transformations périodiques, soit à partir d'un groupe d'ordre un de translations du tore en y ajoutant une, deux ou trois transformations périodiques. Ceux de ces groupes dont les transformations conservent le sens, peuvent être représentés comme des groupes de transformations birationnelles d'une surface de Riemann algébrique de genre un en elle même.

From the author's summary.

de Kerékjártó, B. Sur les groupes intégrables d'ordre trois. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 683–699 (1941). (Hungarian. French summary)

L'auteur démontre les théorèmes suivants. (1) Tout groupe intégrable G_3 d'ordre 3 admet un sous-groupe commutatif invariant G_2 d'ordre 2. Le groupe G_3 est le produit de G_2 et d'un sous-groupe continu d'ordre 1. (2) Tout groupe intégrable d'ordre 3 est homéomorphe à l'un des groupes suivants:

(1) $x'=e^{\gamma}x+\alpha,$	$y'=e^{\gamma}y+\beta,$	$z'=z+\gamma;$
(2) $x'=x+\gamma y+\alpha,$	$y'=y+\beta,$	$z'=z+\gamma;$
(3) $x'=e^{\alpha\gamma}x+e^{\alpha\gamma}\gamma y+\alpha,$	$y'=e^{\alpha\gamma}y+\beta,$	$z'=z+\gamma;$
(4) $x'=e^{(\alpha+1)\gamma}x+\alpha,$	$y'=e^{(\alpha-1)\gamma}y+\beta,$	$z'=z+\gamma;$
(5) $x'=x\cos\gamma-y\sin\gamma+\alpha,$	$y'=x\sin\gamma+y\cos\gamma+\beta,$	$z'=z+\gamma;$
(6) $x'=e^{\alpha\gamma}(x\cos\gamma-y\sin\gamma)+\alpha,$	$y'=e^{\alpha\gamma}(x\sin\gamma+y\cos\gamma)+\beta,$	$z'=z+\gamma;$
(7) $x'=x+\alpha,$	$y'=y+\beta,$	$z'=z+\gamma.$

De cette énumération résulte immédiatement la proposition suivante. (3) Tout groupe non commutatif, intégrable d'ordre 3 peut être représenté comme un groupe d'affinités du plan.

On obtient comme corollaires des théorèmes ci-dessus les caractérisations suivantes du groupe des mouvements du plan euclidien. (4) Si les éléments d'un groupe intégrable d'ordre 3 sont des transformations régulières du plan en lui-même, le groupe est homéomorphe au groupe des mouvements du plan euclidien. (5) Si G_3 est un groupe intégrable d'ordre 3, et si le groupe facteur G_3/G_2, relatif au sous-groupe invariant G_2 d'ordre 2 de G_3, est un groupe cyclique, G_3 est homéomorphe au groupe des paramètres correspondant au groupe des mouvements du plan euclidien.

From the author's summary.

Kiokemeister, Fred. A theory of normality for quasigroups. Amer. J. Math. **70**, 99–106 (1948).

If a quasigroup G has no unit element a normal subquasigroup can no longer be defined as the kernel of a homomorphism. However, every homomorphism of G onto a quasigroup G' does introduce an equivalence relation β into G, elements of G being equivalent under β if and only if they have the same images in G'. The notation $a\beta b$ indicates that a and b are equivalent under β, and the following three laws are satisfied: (i) $ca\beta cb$ implies $a\beta b$; (ii) $ac\beta bc$ implies $a\beta b$; (iii) $a\beta b$ and $c\beta d$ imply $ac\beta bd$. An equivalence relation which satisfies (i), (ii), and (iii) is called a normal relation. The normal relations of G are partially ordered by writing $\beta\geq\alpha$ if and only if aab implies $a\beta b$. Union and crosscut are then defined in the usual way and it is shown that the normal relations of G form a modular lattice.

The set of elements equivalent to a under a normal relation α is denoted by $R(a,\alpha)$. A subquasigroup H of G is called a normal divisor if and only if there exists a normal relation α such that $H=R(a,\alpha)$, where a is any element of H. The relation α is then said to be a normal divisor relation at a. The normal divisor relations at a form a modular lattice which is isomorphic to the lattice of normal divisors of G which contain the element a. The equivalence classes $R(a,\alpha)$ form a quotient quasigroup G/α. The second law of isomorphism is established and the Jordan-Hölder theorem for composition series follows. Applications are made to quasigroups with idempotent elements and to loops.

D. C. Murdoch (Vancouver, B. C.).

Clifford, A. H., and Miller, D. D. Semigroups having zeroid elements. Amer. J. Math. **70**, 117–125 (1948).

The authors consider semigroups S containing zeroid elements, i.e., elements divisible both on the right and the left by every element of S. They prove that a semigroup S contains zeroid elements if and only if both the set of all right ideals and the set of all left ideals of S have nonvoid intersection. If these conditions are satisfied, these intersections both coincide with the set U of all zeroid elements of S. Then U is both an ideal and a subgroup of S. The identity z of U belongs to the center of S. Consequently the mapping ζ, mapping an element x of S onto zx, is a homomorphism of S onto U. The inverse image of z under ζ is a subsemigroup J of S. The set-theoretic union of J and U is also a subsemigroup of S, termed the frame of S. The authors describe a general method of constructing semigroups with a given frame. The paper closes with an account of some properties of the subsemigroups and subgroups of semigroups with zeroid elements and, finally, show that, if S is a semigroup with a homomorphism θ onto a group U, S can be imbedded in a semigroup T with U as its group of zeroid elements in such a way that θ coincides with the homomorphism ζ of T onto U in S. *D. Rees.*

Vorob'ev, N. N. Normal subsystems of finite symmetric associative systems. Doklady Akad. Nauk SSSR (N.S.) **58**, 1877–1879 (1947). (Russian)

The one-to-one maps of any subset of the set $\{1,\cdots,n\}$ onto another subset of this set form an associative system C_n. It is proved that for $n\geqq 5$ the only invariant subsystems [in the sense of E. Liapin, Rec. Math. [Mat. Sbornik] N.S. **20**(62), 497–515 (1947); these Rev. **9**, 134] are the symmetric and the alternating groups S_n and A_n.

S. Eilenberg (New York, N. Y.).

NUMBER THEORY

Sprague, R. Bemerkungen über eine spezielle Abelsche Gruppe. Math. Z. **51**, 82–84 (1947).

The author describes a simple game which has, like Nim, a complete mathematical theory. There is a row of sixteen compartments, five of which are occupied by counters, all alike. Two people play alternately, moving one counter to some unoccupied compartment farther to the left. The final move leaves the five counters in the first five compartments, and the player who makes this move is the winner. In the game of Nim [see W. W. R. Ball, Mathematical Recreations

and Essays, 11th ed., Macmillan, New York, 1947, pp. 36, 73; these Rev. 8, 440] we express certain numbers in the binary scale and then add them as if they were marks of a binary Galois field. Similarly in the new game, the trick is to associate the sixteen compartments with the marks of $GF[2^4]$ in the special order 0, 1, 10, 100, 111, 1000, 1011, 1101, 1110, 1111, 1100, 1010, 1001, 110, 101, 11. When the positions of the five counters correspond to marks whose sum is not zero, the next player can win. His move should be such as to make the sum zero, which can always be done. His opponent cannot avoid changing the sum; so the process continues until the winning move makes the sum $0+1+10+100+111=0$. The author does not tell us whether the chosen order of marks was purely empirical, or whether it has some deeper significance. *H. S. M. Coxeter.*

Sierpiński, W. Remarque sur une hypothèse des Chinois concernant les nombres $(2^n-2)/n$. Colloquium Math. 1, 9 (1947).

The paper deals with the congruence $2^{n-1}\equiv 1 \pmod{n}$, for odd composite values of n. A solution of this congruence, $n=37\cdot 73$, is given in Lucas' Théorie des Nombres [Paris, 1891, p. 422]. Banachiewicz [C. R. Soc. Sci. Lett. Varsovie. Cl. III. 2, 7–10 (1909), and a footnote to the present paper] found seven more solutions: $341=11\cdot 31$, $561=3\cdot 11\cdot 17$, $645=3\cdot 5\cdot 43$, $1105=5\cdot 13\cdot 17$, $1387=19\cdot 73$, $1729=7\cdot 13\cdot 19$ and $1905=3\cdot 5\cdot 127$. Using the fact that, if $2^{n-1}-1$ is divisible by n, then $2^{2^n-2}-1$ is divisible by 2^n-1, the author proves that the congruence has an infinite number of solutions. *A. W. Boldyreff* (Albuquerque, N. M.).

Fine, N. J. Binomial coefficients modulo a prime. Amer. Math. Monthly 54, 589–592 (1947).

The author studies the divisibility of binomial coefficients by a given prime p and proves among other things the following result. A necessary and sufficient condition that none of the binomial coefficients of order M with $M=M_0+M_1p+\cdots+M_kp^k$ $(0\leqq M_r<p;\ M_k>0)$ be divisible by p is that $M_r=p-1$ for $r<k$. *T. Nagell* (Uppsala).

Gloden, A. Sur les nombres terminaux des cubes dans le système de numération décimal. Euclides, Madrid 7, 393–397 (1947).

Tables are given of the 1-, 2-, 3-digit endings of cubes. With each possible ending is given the linear form of those numbers whose cube ends as specified. The 3-digit table contains 505 such entries. *D. H. Lehmer.*

Gloden, A. Parametric solutions of two multi-degreed equalities. Amer. Math. Monthly 55, 86–88 (1948).

Six solutions of $x_1^n+\cdots+x_s^n=y_1^n+\cdots+y_s^n$, $n=2, 4$, in which x_i and y_i are homogeneous forms of the fourth degree in two variables. Each solution provides a solution of $x_1^n+\cdots+x_7^n=y_1^n+\cdots+y_7^n$, $n=1, 2, 4, 6, 8$.
N. G. W. H. Beeger (Amsterdam).

Dorwart, H. L. Note on multi-degree equalities. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. 17, 121–122 (1947).

Consider the four sets of integers (A) 1, 19, 28, 59, 65, 90, 102; (B) 2, 14, 39, 45, 76, 85, 103; (C) 1, 25, 31, 84, 87, 134, 158, 182, 198; (D) 2, 18, 42, 66, 113, 116, 169, 175, 199. Then Escott and Letac have shown that (A) and (B) have the same sums of pth powers for $0\leqq p\leqq 6$, and the same is true of (C) and (D) for $0\leqq p\leqq 8$. This note shows how these results may be derived by successive application of the following theorem. Let $a_1, \cdots, a_r$ and $b_1, \cdots, b_r$ be two sets of integers having the above property for $0\leqq p\leqq n$; then for every k the sets $a_1, \cdots, a_r, b_1+k, \cdots, b_r+k$ and $b_1, \cdots, b_r, a_1+k, \cdots, a_r+k$ have the property for $0\leqq p\leqq n+1$. *D. H. Lehmer* (Berkeley, Calif.).

Reynolds, John O. On the irreducibility of certain polynomials. J. Elisha Mitchell Sci. Soc. 63, 120–132 (1947).

This paper contains various results on the irreducibility of polynomials in the field of rational numbers. The author's main result is the following theorem. If $f(x)$ is a polynomial of degree n with rational integers as coefficients, that takes the value p^k, p a prime, at $t>\frac{1}{2}n$ integral points $a_1, \cdots, a_t$, which are incongruent (mod p), then $f(x)$ is either irreducible in the field of rational numbers or the product of two irreducible factors of equal degree for $n=4, 5, 8, 9, 10$ and 12. *T. Nagell* (Uppsala).

Yakovkin, M. V. Generalization of a criterion of irreducibility of polynomials. Doklady Akad. Nauk SSSR (N.S.) 58, 1915–1918 (1947). (Russian)

The author proves two theorems which generalise a criterion for the irreducibility of a polynomial given by Pólya and already extended by himself [C. R. (Doklady) Acad. Sci. URSS (N.S.) 28, 771–773 (1940); these Rev. 2, 117]. Theorem 1 states that a polynomial $f(x)$ of degree n with integral coefficients is irreducible in the field of the rational numbers if (i) there exists a rational number p/q whose distance from the real part of each root of $f(x)$ exceeds unity, and if (ii) $q^n f(p/q)$ is prime. Theorem 2 concerns the number of factors into which a polynomial can be split, and is a generalisation of the second theorem in the paper referred to above. *R. A. Rankin* (Cambridge, England).

Bambah, R. P., and Chowla, S. On integer roots of the unit matrix. Proc. Nat. Inst. Sci. India 13, 241–246 (1947).

Proof is given for the case $p=3$ of the following conjecture which was announced earlier [Science and Culture 12, 105 (1946); these Rev. 8, 135]. Let X denote any matrix of integers which is a pth root of the identity matrix E_{p-1} of order $p-1$, with $X\neq E_{p-1}$ and p any prime. Then there exists a matrix Δ with integral elements such that $X=\Delta^{-1}M\Delta$, the elements m_{ij} of M being -1 if $i=1$, 1 if $i=j+1$, 0 otherwise. *I. Niven* (Eugene, Ore.).

Bambah, R. P., and Chowla, S. On a function of Ramanujan. Proc. Nat. Inst. Sci. India 12, 433 (1946).

The authors announce the congruences $\tau(n)\equiv\sigma_{11}(n) \pmod{256}$ (n odd), $\tau(n)\equiv 5n^2\sigma_7(n)-4n\sigma_9(n) \pmod{125}$ $(n, 5)=1$; these are proved in the two papers reviewed below. *D. H. Lehmer* (Berkeley, Calif.).

Bambah, R. P., and Chowla, S. The residue of Ramanujan's function $\tau(n)$ to the modulus 2^s. J. London Math. Soc. 22, 140–147 (1947).

The authors prove that if $n=2^m k$, k odd, $m\leqq 0$, then Ramanujan's $\tau(n)$ enjoys the congruence property

$$\tau(n)\equiv(-2)^{3m}(2m+1)\sigma_{11}(k) \pmod{2^8}.$$

Here $\sigma_{11}(k)$ denotes the sum of the eleventh powers of the divisors of k. *D. H. Lehmer* (Berkeley, Calif.).

Chowla, S. A note on multiplicative functions. Proc. Nat. Inst. Sci. India **12**, 429–430 (1946).

A function f is said to be multiplicative if $f(mn)=f(m)f(n)$ whenever m and n are coprime integers. The author defines a function f to be multiplicative mod k if $f(mn)\equiv f(m)f(n)$ (mod k) whenever m, n and k are prime to each other in pairs. This idea is used to prove that

$$\tau(n)\equiv 5n^3\sigma_7(n)-4n\sigma_3(n) \pmod{5^3},$$

where $\tau(n)$ is Ramanujan's function, a multiplicative function. The right side of the above congruence is shown to be multiplicative mod 125. *D. H. Lehmer* (Berkeley, Calif.).

Bambah, R. P., and Chowla, S. A congruence property of Ramanujan's function $\tau(n)$. Proc. Nat. Inst. Sci. India **12**, 431–432 (1946).

The authors give a new proof of the fact that $\tau(n)\equiv\sigma(n)$ (mod 3). The method utilizes the fact that $\tau(n)$ enters into the exact formula for the number of representations of n as the sum of 24 squares. *D. H. Lehmer* (Berkeley, Calif.).

Chowla, S. On a theorem of Walfisz. J. London Math. Soc. **22**, 136–140 (1947).

Let m be any number of the form $m=2^a3^b5^c7^d23^e691^f$, where the exponents are arbitrary nonnegative integers. This note contains the proof of the fact that Ramanujan's function $\tau(n)$ is divisible by m for almost all n. This is made to follow from the lemma: let r and k be arbitrary positive integers; then almost all positive integers contain at least r different primes of the form $kn-1$ each raised to an odd power. The theorem then follows from known congruence properties of $\tau(n)$ with respect to the moduli 2, 3, 5, 7, 23, and 691. *D. H. Lehmer* (Berkeley, Calif.).

Chowla, S. D., and Vijayaraghavan, T. On the largest prime divisors of numbers. J. Indian Math. Soc. (N.S.) **11**, 31–37 (1947).

Let $g(m)$ denote the largest prime divisor of m and let $\phi(x,\theta)$, $0<\theta<1$, denote the number of numbers $m\leqq x$ for which $g(m)>x^\theta$. It is shown that $\lim_{x\to\infty}\phi(x,\theta)/x=A(\theta)$ exists, and that $A(\theta)$ is a strictly decreasing continuous function in the interval $0<\theta<1$, with $A(1-0)=0$, and $A(+0)=1$. *R. Bellman* (Stanford University, Calif.).

Kadian, Sher Singh. Numbers representable by a ternary quadratic form. I. Math. Student **14**, 22–23 (1946).

Chowla, S., and Nazir, Abdur Rahman. Numbers representable by a ternary quadratic form. II. Math. Student **14**, 23 (1946).

The conjecture is made [first note] that positive numbers prime to 11 and congruent to 1 mod 4 are represented by $x^2+y^2+11z^2$. This conjecture is true and was proved in an unpublished manuscript by the reviewer. A similar conjecture [second note] that $x^2+y^2+13z^2$ represents every positive integer congruent to 1 mod 52 will be, in the reviewer's opinion, more difficult to prove. *G. Pall* (Chicago, Ill.).

Bouwkamp, C. J. On the construction of simple perfect squared squares. Nederl. Akad. Wetensch., Proc. **50**, 1296–1299 = Indagationes Math. **9**, 622–625 (1947).

Brooks, R. L., Smith, C. A. B., Stone, A. H., and Tutte, W. T. A simple perfect square. Nederl. Akad. Wetensch., Proc. **50**, 1300–1301 = Indagationes Math. **9**, 626–627 (1947).

Brooks, Smith, Stone and Tutte developed a method of constructing "simple" partitions of a square into mutually different squares [Duke Math. J. **7**, 312–340 (1940); these Rev. **2**, 153]. Bouwkamp criticized their method [same Proc. **50**, 72–78 = Indagationes Math. **9**, 57–63 (1947); these Rev. **8**, 398]. He now retracts his criticism, analyses their example and gives a related one. His note also contains a correction of his list of the different types of squarings of order less than 14.

The second note also deals with Bouwkamp's criticism. The authors give the explicit formula for their example in Bouwkamp's notation. *P. Scherk* (Saskatoon, Sask.).

Koksma, J. F. On Euler's indicator. Nieuw Tijdschr. Wiskunde **35**, 189–195 (1947). (Dutch)

It is proved that for natural numbers a, b and N we have

$$\sum_{n=1}^{N}(am+b)^{-1}\varphi(am+b)=AN+\theta(1+2c+\log(aN+b)),$$

where

$$c=(a,b),\quad A=6\pi^{-2}c^{-1}\varphi(c)\prod_{p\mid a}(1-p^{-2})^{-1},\quad |\theta|\leqq 1.$$

N. G. de Bruijn (Delft).

Ward, Morgan. Memoir on elliptic divisibility sequences. Amer. J. Math. **70**, 31–74 (1948).

An elliptic divisibility sequence is a sequence (h) such that (A) $h_{m+n}h_{m-n}=h_{m+1}h_{m-1}h_n^2-h_{n+1}h_{n-1}h_m^2$ and (B) h_n are integers such that h_n divides h_m whenever n divides m. A detailed study is made of such sequences. Two sequences (h) and (k) are called equivalent if $h_n=c^{n^2-1}k_n$ with $c\neq 0$. Except for certain trivial solutions, it is shown that the solutions of (A) are of one of the forms $h_n=n$, $h_n=\sin n\theta/\sin\theta$, $h_n=\sigma(nu)/\sigma(u)^{n^2}$, $h_n=\lambda_n c^{1-n^2}$, where $\lambda_n=0$ or ± 1 according to the residue of n modulo a fixed odd number. Such topics as the rank of apparition of a prime, periodicity and periodicity and symmetry modulo a prime are discussed. Lucas believed that there is a connection between elliptic functions and certain numerical functions defined by linear recurrence relations of orders three and four. Since the properties found for elliptic divisibility sequences differ from those of solutions of such recurrence relations and since (A) is the fundamental relation of real multiplication theory of elliptic functions, the author concludes that the connection that Lucas claimed cannot be found in the real multiplication theory of elliptic functions. *H. S. Zuckerman.*

Gatteschi, Luigi. Un perfezionamento di un teorema di I. Schur sulla frequenza dei numeri primi. Boll. Un. Mat. Ital. (3) **2**, 123–125 (1947).

It was shown by I. Schur that, for $x\geqq 29$, there is always a prime p satisfying the inequalities $x\leqq p\leqq 5x/4$. The author sharpens this result by reducing the 5/4 to 11/9. The result is obtained by using classical prime number theory for the range $x\geqq 162{,}754$, and then using the tables for the finite range $29\leqq x<162{,}754$. *R. Bellman.*

Selberg, Sigmund. An upper bound for the number of cancelled numbers in the sieve of Eratosthenes. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 2, 3–6 (1946). (Norwegian)

Denote by $P(N)$ the product of all primes not exceeding N and by $A(N_1,N)$ the number of positive integers not exceeding N which are relatively prime to $P(N_1)$. The author proves the existence of an absolute constant c such that $A(N_1,N)<cN/\log N_1$ for all integers N and N_1 with $2\leqq N_1<N$. *T. Nagell* (Uppsala).

Mirsky, L. On coprime values taken by given polynomials. Amer. Math. Monthly 55, 88–89 (1948).

Let $f_1, \cdots, f_s$ be given polynomials with integral coefficients, no r of which possess a nonconstant common factor ($r \leqq s$). An integer is called admissible if no r of the integers possess a common factor greater than 1. Further, denote by $D(p)=D(p; r; f_1, \cdots, f_s)$ the number of integers $1 \leqq x \leqq p$ for which at least r of the s congruences $f_i(x) \equiv 0 \pmod{p}$, $1 \leqq i \leqq s$, are satisfied. The author proves that the admissible numbers form exactly $\prod_{p \mid k}(p-D(p))$ residue classes mod $\prod_{p \mid k} p$, where k is a number depending on $r, s, f_1, \cdots, f_s$. Some applications are given. *P. Erdös.*

Erdös, P. On the integers having exactly K prime factors. Ann. of Math. (2) 49, 53–66 (1948).

Let $\pi_k(n)$ denote the number of integers not exceeding n which have exactly k prime factors, multiple factors being counted only once. The author proves that, for k in the interval $(x-cx^{\frac{1}{2}}, x+cx^{\frac{1}{2}})$, where $x=[\log \log n]$ and c is a constant, $\pi_k(n)=(1+o(1))(n/\log n)x^{k-1}/(k-1)!$. The proof requires estimates for sums such as $\sum 1/a_i^{(k)}$, where the $a_i^{(k)}$ are the integers not exceeding n having exactly k prime factors. At one point only, the prime number theorem is used. The author states two analogous results, one for the case when the multiple factors are counted according to their multiplicity, and the other for the case of square-free integers not exceeding n. He then raises several questions which he is unable to answer at present. *R. D. James.*

Čudakov, N. G. On certain trigonometric sums containing prime numbers. Doklady Akad. Nauk SSSR (N.S.) 58, 1291–1294 (1947). (Russian)

The author estimates certain trigonometrical sums involving primes by relating them to the zeros ($\rho=\beta+i\gamma$) of $\zeta(s)$, in contrast to the method of Vinogradov, which avoids all reference to the ζ-function and the L-functions. The general result is as follows. Suppose $a, b \approx N$ $(0<a<b)$; $f(x)$ real in $[a, b]$; $f'(x) \approx A$, $f''(x) \neq 0$, $(xf'(x))' \approx B$, $(xf'(x))'' \neq 0$, for $a \leqq x \leqq b$, $x \geqq x_0$, where $A, B>0$ and $N^{-1} \ll A \ll N^{-\frac{1}{2}}$, $B \ll 1$. Then

$$(1) \qquad S=\sum_{a \leqq p \leqq b} e^{if(p)} \ll ((N/A)^{\frac{1}{2}}+(N/B)^{\frac{1}{2}}) \log N$$

[$X \ll Y$ means $|X| \leqq c|Y|$ ($c>0$, constant); $X \approx Y$ (not the author's notation) means $X \ll Y \ll X$]. A typical example is as follows. Suppose $b>a \geqq e^2$; $a, b \approx N$; $\frac{1}{4} \leqq \mu \leqq 2$, $\mu \neq 1$; $1 \ll t \ll N^{\frac{1}{2}}(\log N)^{-1}$. Then

$$\sum_{a \leqq p \leqq b} e^{it(\log p)^{\mu}} \ll N(\log N)^{11/8} t^{-\frac{1}{2}} |\mu-1|^{-\frac{1}{2}}.$$

It is not claimed that this method applies to Vinogradov's sums generally. Thus it fails (as it stands) when $f(x)=\alpha x$, a case of the first importance in connection with Goldbach's problem.

Sketch of proof of (1). For $3 \leqq T \leqq a \leqq x$,

$$\psi(x)=\sum_{n \leqq x} \Lambda(n)=x-\sum_{|\gamma| \leqq T} x^{\rho}/\rho+R_x, \qquad R_x \ll T^{-1}x \log^2 x$$

[Landau, Acta Math. 35, 271–294 (1912), Satz 1]; whence

$$(2) \quad \sum_{a \leqq n \leqq b} \frac{\Lambda(n)}{\log n} F(n)=\sum_{a \leqq n \leqq b} \frac{F(n)}{\log n} - \sum_{|\gamma| \leqq T} \sum_{a \leqq n \leqq b} \frac{n^{\rho}-(n-1)^{\rho}}{\rho \log n} F(n)+\sum_{a \leqq n \leqq b} \frac{R_n-R_{n-1}}{\log n} F(n),$$

where $F(x)=e^{if(x)}$. The left hand side is $S+O(N^{\frac{1}{2}})$. On the right hand side the last sum is $O(T^{-1}N^2A \log N)$, by partial summation. The first is $O(A^{-1})$, by a theorem of van der Corput [Landau, Vorlesungen über Zahlentheorie, v. 1, Leipzig, 1927, Satz 352 (and Abel's lemma)]. In the second (double) sum $n^{\rho}-(n-1)^{\rho}$ is replaced by $\rho n^{\rho-1}$, with a total error $O(T^2 \log T)$; $\sum_n$ is then estimated, again by van der Corput's theorem (but less directly), and the (modified) double sum found to be

$$\ll \sum_{|\gamma| \leqq T} \frac{N^{\beta-1}}{\log N}\left\{\frac{N}{B}\right\}^{\frac{1}{2}} \leqq \frac{4}{\sqrt{B}}\left(\frac{N(\frac{1}{2}, T)}{\log N}+\int_{\frac{1}{2}}^{1} N(\sigma, T)N^{\sigma-1}d\sigma\right),$$

where $N(\sigma, T)$ is the number of ρ with $\beta \geqq \sigma$, $0<\gamma \leqq T$. Finally, $N(\sigma, T) \ll T^{\lambda(1-\sigma)}$, $\lambda=8/3$; and (1) follows on taking $T=(NA)^{\frac{1}{2}}$. The last steps are based on an idea of Hoheisel [cf. Ingham, Quart. J. Math., Oxford Ser. 8, 255–266 (1937)].

[Notes by the reviewer. (i) The inequality for $N(\sigma, T)$, attributed to the reviewer, was proved by him [loc. cit., theorem 3] only in the form $N(\sigma, T) \ll T^{\lambda(1-\sigma)} \log^5 T$. But this is true with $\lambda<8/3$; and this, with Čudakov's theorem that $N(\sigma, T)=0$ for $\sigma>1-(\log T)^{-\alpha}$, $T>T_0$, where $0<\alpha<1$, implies the result stated. (ii) The proof of (1) would seem to admit of simplification. Thus (2) may be replaced by

$$\sum_{a<n \leqq b} \frac{\Lambda(n)}{\log n} F(n)=\int_a^b \frac{F(x)}{\log x} d\psi(x)$$

$$=\int_a^b \frac{F(x)}{\log x}dx-\sum_{|\gamma| \leqq T}\int_a^b \frac{F(x)}{\log x} x^{\rho-1}dx+\int_a^b \frac{F(x)}{\log x} dR_x.$$

The last integral on the right may be estimated by partial integration, and the others reduced to integral analogues of van der Corput's sums (more easily estimated than the sums).] *A. E. Ingham* (Cambridge, England).

Linnik, Yu. V., and Ren'i, A. A. On certain hypotheses in the theory of Dirichlet characters. Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 539–546 (1947). (Russian)

[The second author's name appears as Rényi in non-Russian publications.] The authors prove the following lemma. If $0<\eta \leqq 1$, $0 \leqq \vartheta \leqq 1$, $k>1$, $x>1$, $a(m)$ is a multiplicative arithmetical function, $|a(m)| \leqq 1$, and if $f(y)$ is a function with period 1, then

$$|\sum n^{-1}a(n)f(n\vartheta)| \leqq c \max |\sum_{m=1}^{n} m^{-1}a(m)f(m\vartheta)|,$$

where c is a constant depending only on k, where the sum on the left is taken over all $n \leqq \eta x$ whose prime divisors do not exceed $x^{1/k}$, and where the max is taken over the range $0 \leqq \vartheta \leqq 1$, $1 \leqq n \leqq x$. With the help of this lemma, the authors show that, if $\chi(n)$ is a nonprincipal Dirichlet character mod D, and if $\chi(n)=1$ for $1 \leqq n \leqq x^{1/k}$, then

$$(*) \qquad \sum_{n=1}^{x} \chi(n)=o(D^{\frac{1}{2}} \log D)$$

as D tends to infinity uniformly in x. If $\chi(n)$ is real and $\chi(n)=\mu(n)$ for $1 \leqq n \leqq x^{1/k}$, it is also proved that (*) holds with help of Davenport's theorem

$$\sum_{n=1}^{x} \mu(n)e^{2\pi i n\vartheta}=O(x \log^{-k} x)$$

uniformly in ϑ for fixed $k>0$ [Quart. J. Math., Oxford Ser. 8, 313–320 (1937)]. *H. Heilbronn* (Bristol).

Cugiani, Marco. Nuova osservazione sopra un vecchio teorema di Liouville. Boll. Un. Mat. Ital. (3) **2**, 125–128 (1947).

Before the transcendentality of e was demonstrated by Hermite, it was shown by Liouville, using elementary means, that equations of the form $ae^2+be+c=0$, $ae^4+be^2+c=0$, a, b, c rational integers, were impossible. Ricci [same Boll. (2) **13**, 89–92 (1934)] returned to the elementary attack of Liouville and showed that certain polynomial relations in the Gaussian field were impossible. The author continues in this direction, demonstrating that various polynomial relations in the field $K(i\sqrt{3})$ are impossible.

R. Bellman (Stanford University, Calif.).

Davenport, H. On a theorem of Markoff. J. London Math. Soc. **22**, 96–99 (1947).

A simpler proof is given of the following result due to Korkine and Markoff [Markoff, Math. Annalen **56**, 233–251 (1903)]: let $Q(x, y, z)$ be an indefinite ternary quadratic form with real coefficients and with determinant $D\neq 0$; then there exist integers x, y, z (not all zero) for which $|Q(x, y, z)|\leqq(\frac{4}{7}|D|)^{\frac{1}{3}}$. The sign of equality is necessary if and only if Q is equivalent to a multiple of $x^2+y^2-z^2+xz+yz$.

H. S. A. Potter (Aberdeen).

Hallum, Kathleen C., and Mahler, Kurt. On the minimum of a pair of positive definite Hermitian forms. Nieuw Arch. Wiskunde (2) **22**, 324–354 (1948).

Let $f(x, y)=ax\bar{x}+b\bar{x}y+\bar{b}x\bar{y}+cy\bar{y}$ be a positive definite Hermitian form of determinant $ac-b\bar{b}=1$. Two such forms f_1, f_2 have the joint invariant $j=a_1c_2+a_2c_1-b_1\bar{b}_2-\bar{b}_1b_2$, which necessarily satisfies $j\geqq 2$. The problem studied here is the determination of $m(j)$, defined as the least number m such that the simultaneous inequalities $f_1(x, y)\leqq m$, $f_2(x, y)\leqq m$ are soluble in integers x, y (not both zero) of the field $k(i)$, for every pair of forms of invariant j. The problem is approached through the consideration of critical pairs of forms, i.e., pairs for which the above inequalities are not soluble if $m<m(j)$. It is shown that each of the forms of a critical pair is equivalent to a form of a special type. Finally, a rule is obtained for determining $m(j)$, which involves only a finite number of trials for any given value of j. The authors have found $m(j)$ explicitly for $2\leqq j\leqq 6$, and the results [p. 353] show close analogy to Mahler's results for the corresponding problem with two quadratic forms.

H. Davenport (London).

Hinčin, A. Ya. A transfer theorem for singular systems of linear equations. Doklady Akad. Nauk SSSR (N.S.) **59**, 217–218 (1948). (Russian)

The author generalizes his earlier transfer theorem (Übertragungssatz) [Rend. Circ. Mat. Palermo **50**, 170–195 (1926)] to a system of several forms; his result has been anticipated by F. J. Dyson [Proc. London Math. Soc. (2) **49**, 409–420 (1947); these Rev. **9**, 271] who proves more general transfer theorems. In particular, the author's result is obtained from theorem 4 of Dyson on putting $\beta=0$.

K. Mahler (Manchester).

Delone, B. N. An algorithm for the "divided cells" of a lattice. Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 505–538 (1947). (Russian)

"The subject of this paper is a new algorithm and its application to Minkowski's problem of the product of two nonhomogeneous linear forms in two variables" [author's abstract]. The first part of the paper gives a geometrical exposition of the theory of minima of indefinite homogeneous binary quadratic forms, originally due to Markov. In this theory a decisive part is played by the algorithm for the development of a real number in a canonical continued fraction.

The second part of the paper deals with the corresponding nonhomogeneous problem, which is, in geometrical language, to find the minimum of $|xy|$ for the points of a lattice Γ not containing the origin of the (x, y)-plane. A "divided cell" is defined as a fundamental cell of Γ with one vertex in each quadrant. From one divided cell another can in general be derived as follows: the two edges intersecting the x-axis are produced until they cut the y-axis and on each edge is chosen the pair of neighbouring lattice-points straddling the y-axis; the four points thus obtained are the vertices of a divided cell. Given a lattice without points on, or edges parallel to, the coordinate axes, the above algorithm can be applied repeatedly, forwards and backwards, and leads to a series of divided cells continuing indefinitely in both directions. Results: (I) every lattice has at least one divided cell; (II) given any one divided cell, every divided cell can be obtained from it by the algorithm.

To each step in the algorithm corresponds a pair of positive integers, giving the number of lattice-points passed over in the displacement of each of the two edges originally cutting the x-axis, and a sign (+ or −) signifying that the displacement of both is either clockwise or anti-clockwise. The complete algorithm thus defines a series of pairs of positive integers with appropriate signs. Result: (III) given such a series of pairs of integers and signs, there is, with certain exceptions, one essentially unique lattice to which it belongs by the algorithm.

It follows easily from (I) that, if $|xy|$ has a lower bound 1 for all points of Γ, then the determinant of Γ is greater than or equal to 4 (this is Minkowski's theorem). Further results for this type of lattice are as follows. (IV) Every point of Γ such that $|xy|<\frac{4}{3}$ is a vertex of a divided cell, the constant $\frac{4}{3}$ being best possible. (V) For every integer n, there exists a Γ_n of this type with determinant $4(1+n^{-2})^{\frac{1}{2}}$. The Γ_n for different n are equivalent neither to each other nor to the critical lattice Γ_∞ of the Minkowski theorem (which has determinant 4). Result (V) answers negatively the question whether the minimum determinants in the Minkowski theorem have a discrete structure similar to that of the Markov forms. The lattice Γ_n is obtained according to (III) by taking the defining series to be

$$\cdots, nn+, nn-, nn+, nn-, \cdots.$$

The corresponding product of two nonhomogeneous linear forms is $(1+n^2)x(x+1)-y(y+n-1)+\frac{1}{2}n$, which has minimum $\frac{1}{2}n$ and determinant $2(1+n^2)^{\frac{1}{2}}$.

Finally, it is shown that an obvious extension of the notion of "divided cell" to 3 dimensions is unprofitable, since a face-centered cubic lattice exists which possesses no divided cells.

F. J. Dyson (Ithaca, N. Y.).

Cassels, J. W. S. A theorem on star domains. Quart. J. Math., Oxford Ser. **18**, 236–243 (1947).

Let K be a plane star body with a set of continuously changing K-admissible lattices Λ_t, $0\leqq t\leqq 1$, of constant positive determinant δ, which satisfies the following conditions. Each lattice Λ_t contains a parallelogram $OP_tQ_tR_t$, where O is the origin and P_t, Q_t, R_t boundary points of K such that OP_t, OQ_t move monotonically in the same direction while $R_t=Q_t-P_t$ and the triangles OQ_tR_t, $O(-R_t)P_t$ are within K.

For star bodies with such lattice systems, the author shows that $\Delta(K)=\delta$ and that each critical lattice contains at least a point P_i or a point Q_i. The proof is obtained by showing that the critical lattices of K are the critical lattices of a parallelogram the midpoints of whose sides are $\pm P_i$, $\pm Q_i$. The result is applied to determine the critical lattices of some special bounded star bodies. *D. Derry.*

Cassels, J. W. S. The lattice properties of asymmetric hyperbolic regions. I. On a theorem of Khintchine. Proc. Cambridge Philos. Soc. **44**, 1–7 (1948).

Let $k\geqq 0$, $l\geqq 0$; let $K>0$, $L>0$ be such that

$$K+L\geqq\max(k,l);\quad K^2+2(L+l)K+2(L-l)\geqq 1;$$
$$L^2+2(K+k)L+2(K-k)\geqq 1.$$

(I) If $\theta>0$, α are real numbers, θ irrational, $\alpha\neq m-n\theta$ for every pair of integers m, n and if there is an infinity of pairs of positive integers p, q such that (1) $-k<q(p-\theta q)<l$, then there is also an infinity of pairs of positive integers p, q such that (2) $-K<q(p-q\theta-\alpha)<L$. An easy corollary of (I) is (II): Let Λ be a unimodular lattice in the plane, having a point at the origin but having no other point on the y-axis. Suppose that, for each $u>0$, there is a point of Λ in the region $-k<xy<l$, $y>0$, $|x|<u$. Then, to each $v>0$, and to every point V of the plane, there is a point W in the region $-K<xy<L$, $y>0$, $|x|<v$ such that $W-V$ is a point of Λ. Many results about admissible values of k, l in (1) are known; e.g., for every θ we may take $k=l=5^{-\frac{1}{2}}$ and so $K=L=5^{-\frac{1}{2}}$ [Khintchine, Math. Ann. 111, 631–637 (1935); Jogin, Uchenye Zapiski Moskov. Gos. Univ. Matematika **73**, 37–40 (1944); these Rev. **7**, 273]. Asymmetric results (i.e., with $k\neq l$) concerning (1) are due to B. Segre [Duke Math. J. **12**, 337–365 (1945); these Rev. **6**, 258]. *V. Jarník* (Prague).

***Mullender, Pieter. Toepassing van de Meetkunde der Getallen op Ongelijkheden in $K(1)$ en $K(i\sqrt{m})$. [Application of the Geometry of Numbers to Inequalities in $K(1)$ and $K(i\sqrt{m})$].** Thesis, Free University of Amsterdam, 1945. x+85 pp.

Using Blichfeldt's principle in the geometry of numbers [Trans. Amer. Math. Soc. **15**, 227–235 (1914)] the reviewer and Meulenbeld have deduced a number of theorems on the Diophantine approximations of systems of linear forms [Nederl. Akad. Wetensch., Proc. **44**, 62–74, 426–434 (1941); **45**, 256–262, 354–359, 471–478, 578–584 (1942); these Rev. **2**, 253; **3**, 71; **5**, 256]. Simplifying their method and generalizing their results the author gives a systematic survey, adding various new theorems especially on systems of forms in certain complex algebraic fields. He also generalizes a theorem of van der Corput and Schaake [Acta Arith. **2**, 152–160 (1936)]. The first part of the author's paper reviewed below gives an English survey of this thesis. *J. F. Koksma* (Amsterdam).

Mullender, P. Homogeneous approximations. Nederl. Akad. Wetensch., Proc. **50**, 173–185 = Indagationes Math. **9**, 136–148 (1947).

Let $L_k=a_{k1}z_1+\cdots+a_{kn}z_n$ $(k=1,\cdots,n)$ be n linear forms of determinant $\Delta\neq 0$. Let either $\omega=1$ ("real case") or $\omega=im^{\frac{1}{2}}$, where $m>0$ is an integer without quadratic divisors ("complex case"). A point $[z_1,\cdots,z_n]$ is called a lattice point if $z_1,\cdots,z_n$ are integers of the field $K(\omega)$. In the real case it is supposed that the eventual nonreal forms L_k appear in pairs of conjugate complex forms. (I) The first part contains a summary (without proofs) of the author's dissertation [cf. the preceding review]. Let $p>0$, $q>0$ be integers, $p+q=n$, $\sigma>0$;

$$\Lambda_{p,q,\sigma}=q\cdot\sigma^{-1}\cdot(p+q)^{(p+q)/\sigma}(2p)^{-p/\sigma}(2q)^{-q/\sigma}$$
$$\times\int_0^{2q/(p+q)}(1-x)^{p/\sigma}x^{q/\sigma-1}dx$$
$$+p\cdot\sigma^{-1}\int_{2q/(p+q)}^{1}(1-x)^{p/\sigma}x^{-1}dx$$

for $p\geqq q$, $\Lambda_{p,q,\sigma}=\Lambda_{q,p,\sigma}$ for $p\leqq q$; $\zeta^{(\omega)}=1$ for $\omega=1$, $\zeta^{(\omega)}=\frac{1}{4}\pi\lambda m^{-\frac{1}{2}}$ for $\omega=im^{\frac{1}{2}}$; $\lambda=1$ for $m\not\equiv 3\pmod 4$, otherwise $\lambda=2$; $\gamma_{p,q}^{(\text{imb})}=(\zeta^{(\omega)})^{\frac{1}{2}(p+q)}2^{p+q}(\Lambda_{p,q,1})^{\frac{1}{2}}$, $\gamma_{p,q}^{(1)}=2^{p+q}\Lambda_{p,q,1}$; $\sigma'=\sigma$ for $\omega=1$, $\sigma'=\frac{1}{2}\sigma$ for $\omega=im^{\frac{1}{2}}$;

$$\rho_{p,q,\sigma}^{(\omega)}=(\zeta^{(\omega)})^{\sigma'(p+q)}2^{p+q}\left(\frac{(\Gamma(1+1/\sigma'))^{p+q}\Lambda_{p,q,\sigma'}}{\Gamma(1+p/\sigma')\Gamma(1+q/\sigma')}\right)^{\sigma'}.$$

Theorem: let $\sigma\geqq 1$,

$$P=|L_1|^\sigma+\cdots+|L_p|^\sigma,\quad Q=|L_{p+1}|^\sigma+\cdots+|L_{p+q}|^\sigma;$$

then to every $t>0$ there is at least one lattice point $[z_1,\cdots,z_n]\neq[0,\cdots,0]$ such that

$$Q\leqq 2/t,\quad P\leqq 2(t^q|\Delta|^\sigma/\rho_{p,q,\sigma}^{(\omega)})^{1/p},\quad P^pQ^q\leqq|\Delta|^\sigma/\rho_{p,q,\sigma}^{(\omega)};$$

and, to every $\epsilon>0$, there is an infinity of lattice points satisfying $Q<\epsilon$, $P^pQ^q\leqq|\Delta|^\sigma/\rho_{p,q,\sigma}^{(\omega)}$. There is a completely analogous theorem for $P'=\max(|L_1|,\cdots,|L_p|)$, $Q'=\max(|L_{p+1}|,\cdots,|L_{p+q}|)$, only with $|\Delta|$, $\gamma_{p,q}$ instead of $|\Delta|^\sigma$, $\rho_{p,q,\sigma}^{(\omega)}$. Further, extending a result of van der Corput and Schaake to the complex case, the author proves: if $\sigma\geqq 2$, $\omega=im^{\frac{1}{2}}$, there is a lattice point $[z_1,\cdots,z_n]\neq[0,\cdots,0]$ such that $|L_1|^\sigma+\cdots+|L_n|^\sigma\leqq|\Delta|^{\sigma/n}/B$, if

$$0<B<\frac{2(\zeta^{(\omega)})^{\frac{1}{2}\sigma}}{(1+2n/\sigma)^{\sigma/(2n)}}\cdot\frac{(\Gamma(1+2/\sigma))^{\sigma/2}}{(\Gamma(1+2n/\sigma))^{\sigma/(2n)}}.$$

(II) The second part contains the proof of the following theorem. Let $n=3$; we call $A>0$ an allowed value in the inequality $|L_1L_2L_3|\leqq|\Delta|/A$ if this inequality is satisfied by an infinity of lattice points, $[z_1z_2,z_3]$, whatever the coefficients a_{ik} may be (with the obvious condition in the "real case"). Let $C^{(\omega)}$ be the upper bound of all allowed values. Let $C_{2,1}^{(\omega)}$ have an analogous meaning with regard to the inequality $|\alpha_1x_1+\alpha_2x_2-x_3|<1/(AX^2)$, where $X=\max(|x_1|,|x_2|)\geqq 1$. Then $C_{2,1}^{(\omega)}\leqq C^{(\omega)}$. It is easily seen that, if the factorization in prime numbers in $K(\omega)$ is unambiguous, we have $C^{(\omega)}\leqq(D_3^{(\omega)})^{\frac{1}{2}}$, where $D_3^{(\omega)}$ denotes the minimum of the absolute values of the discriminants of all algebraic fields of degree 3 relative to $K(\omega)$, e.g., $D_3^{(1)}=23$ [Furtwängler], and so $C_{2,1}^{(1)}\leqq 23^{\frac{1}{2}}$. *V. Jarník* (Prague).

Steinhaus, H. Sur un théorème de M. V. Jarník. Colloquium Math. **1**, 1–5 (1947).

The author gives a proof of the following theorem, stated and proved by V. Jarník in a recent letter to him. Let J be a closed rectifiable Jordan curve of length l, enclosing an area a. Then, provided $l\geqq 1$, the number w of points with integral coordinates inside J satisfies $|w-a|<l$. The proof is elementary and depends on the following result: if a Jordan arc S joins two points on the boundary of the square $|x|<\frac{1}{2}$, $|y|<\frac{1}{2}$, dividing the square into two regions, and if Δ is the region which does not contain the origin, then the area of Δ is less than the length of S. This is proved by simple considerations in each of four possible cases. *H. Davenport* (London).

Tihanyi, Nikolaus. Die Struktur des Weber-schen Resolventen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **60**, 92–98 (1941). (Hungarian. German summary)

The Weberian resolvent, i.e., the Lagrangian resolvent of the cyclotomic field of order 2^n, $n \geqq 5$ [see H. Weber, Lehrbuch der Algebra, 2d ed., Vieweg, Braunschweig, 1899, v. 2, p. 72] is transformed by means of elementary calculations and expressed by a simple formula. *A. Rényi* (Budapest).

Tihanyi, N. Die Berechnung der Weberschen Resolvente. Mat. Fiz. Lapok **49**, 70–72 (1942). (Hungarian. German summary)

The formula given in the paper reviewed above for the Weberian resolvent of the cyclotomic field of order 2^n is simplified in the case $n = 2k-1$, $k \geqq 3$. *A. Rényi.*

Tihanyi, Nikolaus. Die Berechnung der Lagrangeschen Resolvente in gewissen Kreiskörpern. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **61**, 830–834 (1942). (Hungarian. German summary)

As a generalization of the result of a preceding paper [see the second preceding review] the Lagrangian resolvent of the cyclotomic field of order p^n, where p is an odd prime, $n \geqq 2$, is calculated. *A. Rényi* (Budapest).

Gupta, H. On nth-power residues. Quart. J. Math., Oxford Ser. **18**, 253–256 (1947).

The author gives certain identical congruences, analogous to Bauer's identical congruence, connected with the set of nth power residues of m which are less than and prime to m. *W. H. Simons* (Vancouver, B. C.).

Reuvecamp, W. J., Jr. Eichenberg's theorem. Nieuw Tijdschr. Wiskunde **35**, 202–207 (1947). (Dutch)

Translation of Epstein's exposition [Weber-Wellstein, Enzyklopädie der Elementarmath., 5th ed., Teubner, Leipzig-Berlin, 1934, p. 280]. *N. G. de Bruijn* (Delft).

Beeger, N. G. W. H. A problem of the theory of numbers and its history. Nieuw Arch. Wiskunde (2) **22**, 306–309 (1948).

The field $R(\zeta)$, $\zeta^4 = -1$, is Euclidean; its fundamental unit may be taken as $\epsilon = 1+\zeta-\zeta^3$. The octavic reciprocity law $(a/\pi) = (\pi/a)$, where a is rational and π is a prime in the field, is known to hold. The problem proposed is to prove the following criterion using only the above results: Let $2^{(p-1)/8} \equiv 1 \pmod{p}$, $p = 16n+1 = a^2+b^2 = c^2+2d^2$, $a \equiv c \equiv 1 \pmod 4$; then $2^{(p-1)/16} \equiv (-1)^{b/16+d/4} \pmod{p}$. This criterion was stated without proof by Cunningham [Proc. London Math. Soc. (1) **27**, 85–122 (1896)]. In the present note it is shown that it can be deduced from results of F. Goldscheider [Das reciprocitätsgesetz der achten Potenzreste, Wissensch. Beitr. z. Progr. d. Luisenstädtischen Realgymn. Berlin, 1889]. Unfortunately the results of Goldscheider are not proved in full but the author believes it should be possible to prove them using only the theory mentioned at the beginning of this review. *L. Carlitz* (Durham, N. C.).

Rédei, Ladislaus. Eine Anwendung der hypergeometrischen Reihen auf eine Faktorenzerlegung des Fermatschen Polynoms $1-x^{p-1}$ im Zusammenhang mit der Theorie der quadratischen Reste. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 335–348 (1943). (Hungarian. German summary)

This paper is concerned with the problem of the distribution of quadratic residues. For p an odd prime the author defines four classes of integers $H_{\rho\sigma}$ ($\rho^2 = \sigma^2 = 1$) as consisting of those integers a for which

$$\left(\frac{a-1}{p}\right) = \rho, \quad \left(\frac{a}{p}\right) = \sigma, \qquad 2 \leqq a \leqq p-1,$$

where these are Legendre symbols. Denote by $f_{\rho\sigma}(x)$ the polynomial whose roots are the members of $H_{\rho\sigma}$, without repetition. By a theorem of Lagrange the degree of $f_{\rho\sigma}$ is $d_{\rho\sigma} = [\frac{1}{4}(p-\rho\sigma-\rho-1)]$. The author considers the problem of finding the coefficients of $f_{\rho\sigma}(x)$ modulo p. If $F(\alpha, \beta, \gamma; x)$ denotes the hypergeometric function the author proves that $f_{\rho\sigma}(x)$ is congruent modulo p to the polynomial of degree $d_{\rho\sigma}$ obtained by truncating the function

$$F(\tfrac{1}{4}(\sigma+2), \tfrac{1}{4}(2\rho\sigma+2\rho+\sigma), 1+\tfrac{1}{2}\rho; x).$$

Hence the elementary symmetric functions of the members of the class $H_{\rho\sigma}$ are readily obtained modulo p. *D. H. Lehmer* (Berkeley, Calif.).

Rédei, Ladislaus. Über den geraden Teil der Ringklassengruppe quadratischer Zahlkörper, die Pellsche Gleichung und die Diophantische Gleichung $rx^2+sy^2 = z^n$. I. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 13–34 (1943). (Hungarian. German summary)

Rédei, Ladislaus. Über den geraden Teil der Ringklassengruppe quadratischer Zahlkörper, die Pellsche Gleichung und die Diophantische Gleichung $rx^2+sy^2 = z^n$. II. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 35–47 (1943). (Hungarian. German summary)

Rédei, Ladislaus. Über den geraden Teil der Ringklassengruppe quadratischer Zahlkörper, die Pellsche Gleichung und die Diophantische Gleichung $rx^2+sy^2 = z^n$. III. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **62**, 48–62 (1943). (Hungarian. German summary)

As an application of the results obtained by the author in two previous papers [same Anz. **59**, 829–841 (1940); J. Reine Angew. Math. **186**, 80–90 (1944); these Rev. **7**, 369, 111] regarding the structure of the group of ideal classes (in the widest sense) of an algebraic number field of finite order, the quadratic number field $k = R(d^{\frac{1}{2}})$ is considered and the even part (i.e., the invariants 2^n) of the group of ring classes is investigated. As a result of these investigations a complete survey is given of the existence of solutions of Pell's equation; furthermore, the Diophantine equations $rx^2+sy^2 = 1$ and $rx^2+sy^2 = 2$, called the equations of Dirichlet (the first of which can be considered as a generalization of Pell's equation), and the equations $rx^2+sy^2 = z^n$, $rx^2+sy^2 = 2z^n$ are also investigated (r and s denote relatively prime integers). Part II contains the investigation of the (ring-) class fields K_n for which K_n/k is cyclic of order 2^n; the results of H. Reichardt [J. Reine Angew. Math. **170**, 75–82 (1933)] are generalized. Part III contains the construction of the class fields K_n; a mistake contained in the paper of Reichardt cited above is corrected. A fourth part and a summarizing paper in German are to appear later. *A. Rényi* (Budapest).

Fuchs, Ladislas. A theorem on the relative norm of an ideal. Comment. Math. Helv. **21**, 29–43 (1948).

The following theorem, which holds for algebraic extension fields of the rational numbers, is generalised to relative fields. Assume that the field is of degree n, that the unit ideal is $\mathfrak{o} = (\omega_1, \cdots, \omega_n)$ and that $\mathfrak{a} = (\alpha_1, \cdots, \alpha_n)$ is an arbitrary ideal. It is known that $|\alpha_i^{(k)}|^2 = \text{Norm } \mathfrak{a}^2 |\omega_i^{(k)}|^2$, where the superscripts denote the conjugates. The difficulty in

generalising this theorem lies in the fact that in relative fields ideals may have a basis containing more elements than the relative degree. To overcome this difficulty the concept of an array ideal $((a_{ik}))$ is introduced. The base elements of this ideal are the determinants of order n of a matrix (a_{ik}), $i=1, \cdots, n$; $k=1, \cdots, n$ with $N \geqq n$. To every ideal $\mathfrak{a}=(\alpha_1, \cdots)$ corresponds the array ideal $\mathfrak{a}^*=((\alpha_i^{(k)}))$. It is shown that the relative norm of $\mathfrak{a}$ is $\mathfrak{a}^*/\mathfrak{o}^*$. The square of $\mathfrak{o}^*$ is usually called the relative discriminant of the field, which is an analogous generalisation of the discriminant $|\omega_i^{(k)}|^2$ in the case the base field is the rational numbers.

O. Todd-Taussky (Los Angeles, Calif.).

Hasse, Helmut. Über die Klassenzahl abelscher Zahlkörper. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 126–130 (1946).

The author studies the number of classes of ideals of an absolutely Abelian field K. The well-known formula for the class number h is obtained by analytical methods. It cannot be seen directly that the expression for h represents a positive rational integer. In order to reach an insight into the significance of the formula, it seems necessary to change the expression for h in such a manner that the rational integral character of h becomes obvious from the formula. The class number h of the absolutely Abelian field K is the product h_0h^* of the class number h_0 of the greatest real subfield K_0 and of the relative class number h^* of K with regard to K_0. This number h^* is the number of classes of K for which the relative norm lies in the unit class of K_0. The problem for h then reduces to the corresponding problems for h_0 and for h^*. The results of the author concerning h_0 have been reviewed in connection with another paper [Ber. Math. Tagung Tübingen 1946, pp. 74–75 (1947); these Rev. **9**, 175]. A transformation for the expression for h^* is given which sets in evidence that h^* is a rational integer; the results apply to every absolutely Abelian field. The formulas can be used to obtain generalizations of results of H. Weber. Necessary and sufficient conditions are given that the number of genera of an imaginary Abelian field is an odd number. In the case of an imaginary cyclic field K, the number h is odd if and only if h^* is odd. The present paper does not contain any proofs; the proofs will be given in a later publication. *R. Brauer* (Ann Arbor, Mich.).

Carlitz, L. The singular series for sums of squares of polynomials. Duke Math. J. **14**, 1105–1120 (1947).

The author defines a singular series $S(F; k, s)$ [cf. Hardy, Trans. Amer. Math. Soc. **21**, 255–284 (1920); Estermann, Acta Arith. **2**, 47–79 (1936)] for representations as sums of squares (*) $F=\alpha_1U_1^2+\cdots+\alpha_sU_s^2$, where $\alpha_1, \cdots, \alpha_s$ are nonzero elements of the field $GF(p^n)$, F is an element of the polynomial ring $GF[p^n, x]$, deg $F \leqq 2k$, and the first $l(\geqq 1)$ U's are primary of degree k, while the remaining U's are arbitrary of degree less than k. The series $S(F; k, s)$ is a finite sum involving Gauss sums of roots of unity. Its value is proved to be the number of solutions of (*) for all $s \geqq 2$, F arbitrary, and also for $s=1$ in case deg $F=2k$ and the leading coefficient of F is a square in $GF(p^n)$. The equivalence of this with earlier formulas of the author for numbers of solutions of (*) is proved and relations with more recent results [Cohen, Duke Math. J. **14**, 251–267, 543–557 (1947); these Rev. **9**, 81, 176] are shown. *R. Hull.*

Carlitz, L. Representations of arithmetic functions in $GF[p^n, x]$. Duke Math. J. **14**, 1121–1137 (1947).

An arithmetic function here means a single-valued function $f(A)$, $A \epsilon GF[p^n, x]$, with complex values. An equivalence relation $f \sim g$, meaning $f(A)=g(A)$ for all A such that deg $A<r$, leads to classes of equivalent functions which form rings with respect to ordinary addition and each of the three Cauchy multiplications [L. Carlitz and E. Cohen, same J. **14**, 707–722 (1947); these Rev. **9**, 176]. Sets of functions forming special bases for these rings are determined. Relations between the sets, the Gauss sums and singular series [see the preceding review], and representations by quadratic forms are proved. *R. Hull.*

Carlitz, L. A problem of Dickson's. Duke Math. J. **14**, 1139–1140 (1947).

The author proves the following theorem. Let $F(a)$ be a nonzero square in $GF(p^n)$ for all a in the field, where $F(x)$ is a polynomial of degree k. Then if $p^n>(k-1)^2$, $F=G^2$, where G is a polynomial with coefficients in $GF(p^n)$. In the proof, he employs the Artin zeta-function [Math. Z. **19**, 207–246 (1924)] and the Riemann hypothesis for it, proved by Weil [Proc. Nat. Acad. Sci. U. S. A. **27**, 345–347 (1941); these Rev. **2**, 345]. The result was conjectured by Dickson, and proved by him in a somewhat stronger form when $k=4$ or 6 [Trans. Amer. Math. Soc. **10**, 109–122 (1909)].

R. Hull (Lafayette, Ind.).

ANALYSIS

Borel, Émile. Sur les systèmes généraux de numération. C. R. Acad. Sci. Paris **226**, 1405–1407 (1948).

The author discusses various methods of representing real numbers. A general system can be defined by means of a sequence of rational numbers u_n for which $u_n \downarrow 0$ and $u_n-u_{n+1} \downarrow 0$. For any sufficiently small $a>0$ one can find an n such that $u_{n-1} \geqq a > u_n$. Putting $a=u_n+a_1$ one obtains an algorithm representing a. *W. Feller* (Ithaca, N. Y.).

Horváth, Jean. Sur le rapport entre les systèmes de postulats caractérisant les valeurs moyennes quasi arithmétiques symétriques. C. R. Acad. Sci. Paris **225**, 1256–1257 (1947).

The author shows that the theorem of Aczel [Norske Vid. Selsk. Forh., Trondhjem **19**, no. 23, 83–86 (1947); these Rev. **8**, 504] characterizing mean-value functions of the form $m(\sum_{j=1}^n x_j)=f^{-1}[n^{-1}\sum_{j=1}^n f(x_j)]$, where f is continuous and increasing, can be deduced from a slight modification of a theorem of Kolmogoroff and Nagumo. Simple derivations of both theorems are given.

E. F. Beckenbach (Los Angeles, Calif.).

†**Schoenberg, I. J. Some analytical aspects of the problem of smoothing.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 351–370. Interscience Publishers, Inc., New York, 1948. $5.50.

This note consists of two independent parts. In the first, an expression is obtained for the limiting form of the coefficients in the linear transformation $y_m^{(n)}=\sum_{\nu=-\infty}^{\infty} L_\nu^{(n)} y_{m-\nu}$, which is the n-fold iterate of the smoothing formula $y_m'=\sum_\nu L_\nu y_{m-\nu}$. In the second part, the author obtains a necessary and sufficient condition that a transformation of this form be variation-diminishing, that is, having the prop-

erty that the number of variations of sign in the sequence $\{y'_n\}$ does not exceed the number in the sequence $\{y_n\}$.

T. N. E. Greville (Washington, D. C.).

Verblunsky, S. On the moments of a concave function. J. London Math. Soc. **22**, 115–120 (1947).

(I) If (1) $c_j=\int_0^1 x^j d\phi(x)$, $j=0, 1, \cdots$, where $\phi(x)$ is nondecreasing and bounded, and if $(j+1)(j+2)d_j=\sum_0^j c_k$, then (2) $d_j=\int_0^1 x^j\psi(x)dx$, where $\psi(x)$ is concave and bounded. (II) Conversely, if the numbers d_j are given by (2) where $\psi(x)$ is concave and positive and if $c_0=2d_0$, $c_j=(j+1)(j+2)d_j-j(j+1)d_{j-1}$ for $j=1, 2, \cdots$, then (1) holds, where $\phi(x)$ is nondecreasing and bounded. As a special case of (II) the author obtains a theorem of Blaschke and Pick [Math. Ann. **77**, 277–300 (1916), p. 290] concerning the existence of a bounded nondecreasing solution $\phi(x)$ of the integral equation $\psi(x)=\int_0^1 K(x, t)d\phi(t)$, when $\psi(x)$ is concave and positive and $K(x, t)=xt^{-1}$ or $(1-x)(1-t)^{-1}$ according as $x\leqq t\leqq 1$ or $0\leqq t\leqq x$. *E. Hille.*

Verblunsky, S. On a class of cubics. J. London Math. Soc. **22**, 120–124 (1947).

The author obtains the parametric representation of all polynomials $f(x)$ of degree at most 3 having real coefficients and satisfying $|f(x)|\leqq 1$ for $-1\leqq x\leqq 1$. These cubics may be obtained by a specified linear transformation from three basic polynomials which are given explicitly. The quadratic case is used in a paper by M. R. Rees [see the following review]. *E. Hille* (New Haven, Conn.).

Rees, M. R. A theorem of three moments. J. London Math. Soc. **22**, 124–130 (1947).

Using the results of S. Verblunsky [see the preceding review] the author finds necessary and sufficient conditions for three numbers c_0, c_1, c_2 to satisfy the equations $c_k=\int_{-1}^1 x^k dg(x)$, $k=0, 1, 2$, where $g(x)$ is a real function whose total variation in $[-1, 1]$ does not exceed one. Five inequalities are involved. *E. Hille* (New Haven, Conn.).

Chuang, Chi-Tai. Sur les fonctions continues monotones. Ann. Sci. École Norm. Sup. (3) **64** (1947), 179–196 (1948).

In the first part of this paper the author obtains the following strengthened form of a lemma on monotonic continuous functions due to Borel [Acta Math. **20**, 357–396 (1897), in particular, pp. 375–377]. Suppose that $f(x)$ is continuous, positive and increasing for $x\geqq 0$, $\gamma(x)$ and $\delta(x)$ are continuous positive and decreasing for $x\geqq 0$, and that $\int_0^\infty \gamma(x)dx=M<\infty$. Let E be the set on which $f(x+\gamma[f(x)])-f(x)\geqq\delta(x)$. Then

$$\int_E \delta(x)dx\leqq\gamma(0)\delta(0)+M.$$

The second half of the paper deals with the class of functions for which the set E defined above is void; it is proved that any continuous positive increasing function lies between two functions of this class. Similar results are also proved for decreasing functions. *R. C. Buck* (Providence, R. I.).

Copson, E. T. On the Riesz-Riemann-Liouville integral. Proc. Edinburgh Math. Soc. (2) **8**, 25–36 (1947).

Consider the Riesz integral (2)

$$I^\alpha f(T, X_1, \cdots, X_n)=$$

$$H(\alpha)\int_V f(t, x_1, \cdots, x_n)\{(T-t)^2-\sum(X_i-x_i)^2\}^{\alpha-n}dV.$$

Here V is the region bounded by the retrograde characteristic cone with vertex at $P=(T, X)$ and a space-like surface S. The author shows that, if f and S are of class $C^{[\frac{n}{2}]-1}$ or $C^{[\frac{n}{2}]}$ depending on whether n is even or odd, $\lim_{\alpha\to 0+} I^\alpha(f(P))=f(P)$. The argument depends on a complicated lemma which is at heart a continued Green's theorem reduction. The author considers his method to be simpler than that of Fremberg [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **15**, no. 27, 265–276 (1945); these Rev. **7**, 384], who has treated the same problem. *D. G. Bourgin.*

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, Wacław. Démonstration de l'égalité $2^m-m=2^m$ pour les nombres cardinaux transfinis. Fund. Math. **34**, 113–118 (1947).

Sierpiński, Wacław. Sur la différence de deux nombres cardinaux. Fund. Math. **34**, 119–126 (1947).

If $\mathfrak{m}$ and $\mathfrak{n}$ are two cardinals and if there exists a unique cardinal $\mathfrak{p}$ such that $\mathfrak{m}=\mathfrak{n}+\mathfrak{p}$, then one writes $\mathfrak{p}=\mathfrak{m}-\mathfrak{n}$. The author proves without the use of the axiom of choice various theorems concerning the subtraction of cardinals. The above definition is due to A. Tarski who has also announced, without proofs, most of the theorems proved by the author [Lindenbaum and Tarski, C. R. Soc. Sci. Lett. Varsovie. Cl. III. **19**, 299–330 (1926)].

B. Jónsson (Providence, R. I.).

Sierpiński, Wacław. Sur l'implication $(2\mathfrak{m}\leqq 2\mathfrak{n})\to(\mathfrak{m}\leqq\mathfrak{n})$ pour les nombres cardinaux. Fund. Math. **34**, 148–154 (1947).

The author proves, without using the axiom of choice, that if $\mathfrak{m}$ and $\mathfrak{n}$ are two cardinals such that $2\mathfrak{m}\leqq 2\mathfrak{n}$, then $\mathfrak{m}\leqq\mathfrak{n}$. This theorem was announced without proof by A. Tarski [Lindenbaum and Tarski, C. R. Soc. Sci. Lett. Varsovie. Cl. III. **19**, 299–330 (1926)]. *B. Jónsson.*

Sierpiński, Wacław. Sur une proposition qui entraîne l'existence des ensembles non mesurables. Fund. Math. **34**, 157–162 (1947).

The main concern of the author is to obtain certain known results without the use of the axiom of choice. He starts with proposition P: A "one-valued" image B of a set A (i.e., each element of A has a unique mate in B) is not of higher cardinal than A. On the basis of P, he proves successively, without resort to the axiom of choice, that: (1) $\aleph_1\leqq 2^{\aleph_0}$; (2) if $2^{\aleph_0}$ is the sum of 2 cardinals, at least one of them equals $2^{\aleph_0}$; (3) there exists a linear nondenumerable set containing no (nonnull) perfect set; (4) there exists a linear (Lebesgue) nonmeasurable set. The reasoning utilizes Lebesgue's decomposition of an interval into $\aleph_1$ (nonnull) disjoint sets [J. Math. Pures Appl. (6) **1**, 139–216 (1905), p. 213]; and the author's result [Mathematica, Cluj **3**, 30–32 (1930)] that if we can determine a linear order for the "Vitali classes" V (of real numbers), then we can define a (Lebesgue) nonmeasurable set; here V is a set consisting of all the real numbers which differ from a fixed real number by a rational amount. It is additionally shown that the axiom of choice is equivalent to the following proposition: of two sets, at least one is a one-valued image of the other.

H. Blumberg (Columbus, Ohio).

Lyapunov, A. On R-sets. Doklady Akad. Nauk SSSR (N.S.) **58**, 1887–1890 (1947). (Russian)

The author presents certain generalizations of the operation A, and announces theorems concerning the preservation of measurability and the property of Baire under these generalized operations. Close connections evidently exist with work of Kolmogoroff [Kantorovitch and Livenson, Fund. Math. **20**, 54–97 (1933)]. *E. Hewitt.*

Kronrod, A. S., and Landis, E. M. On level sets of a function of several variables. Doklady Akad. Nauk SSSR (N.S.) **58**, 1269–1272 (1947). (Russian)

This note is concerned with topological and differential properties of the level sets $E_t=\{\xi \mid F(\xi)=t\}$ of a continuous real-valued function F of n real variables $\xi=(x_1, \cdots, x_n)$, especially properties which hold for almost all values of t. Let F be defined on the n-dimensional cube L and have partial derivatives of all orders not exceeding n at every point ξ of L. Let M be the set of points of L where all the first partial derivatives of F equal zero. Then M is disjoint from almost all level sets of F in the sense that $\{F(\xi) \mid \xi \varepsilon M\}$ is of Lebesgue measure zero. Under the same hypotheses on F, almost every level set E_t consists of a finite number of components, each an n-times differentiable (at each point) manifold without boundary or with boundary on the boundary of the cube of definition. Let F be $(n-1)$-times differentiable on the n-dimensional cube L. Then almost every level set E_t has only locally-connected components. The author complements these theorems with the remarks that for $n \geqq 2$ there exist $(n-1)$-times differentiable functions on the n-cube such that the components are infinite in number and not differentiable; for $n>2$ the components need not even be manifolds. Also for $n \geqq 2$ there exist $(n-2)$-times differentiable functions on the n-cube which do not have locally-connected components.

M. M. Day (Urbana, Ill.).

Borel, Émile. Sur les sommes de développements unitaires normaux. C. R. Acad. Sci. Paris **226**, 365 (1948).

Consider the normal expansions defined by the author [same C. R. **225**, 51 (1947); cf. also P. Lévy, ibid., 918–919 (1947); these Rev. **9**, 292]. Let E be the set of numbers for which $a_n \leqq 3a_{n-1}$ for every n. It is stated without proof that E is of measure zero, but that every positive number $x<1$ is the sum of two elements of E. *W. Feller.*

Korovkin, P. P. Generalization of a theorem of D. F. Egorov. Doklady Akad. Nauk SSSR (N.S.) **58**, 1265–1267 (1947). (Russian)

The theorem of Egorov asserts that, if a sequence of measurable functions converges in a set of E of positive and finite measure, then it converges uniformly in a subset of E with measure arbitrarily close to that of E. Here "measure" means a nonnegative and absolutely additive set function. The author points out that the additive character of measure is not indispensable and gives a proof (not essentially different from the usual one) under the condition that the measure, defined for a family $\mathfrak{M}_x$ of sets, is nonnegative, monotone, and such that for a monotone sequence of sets the limit of the measures equals the measure of the limit. Measurability of the function $f(x)$ means that, for every a, the set of points where $f(x) \geqq a$ belongs to $\mathfrak{M}_x$. The author stresses the fact that the capacity of sets is a nonadditive measure satisfying the required conditions. Simple applications to capacity are given. *A. Zygmund.*

Korovkin, P. P. The sets of convergence of series of polynomials. Doklady Akad. Nauk SSSR (N.S.) **58**, 1589–1591 (1947). (Russian)

Given any closed and bounded set F in the z-plane, let $\tau(F)$ denote the capacity of F. For an arbitrary set E we write $\tau_*(E)=\sup \tau(F)$ for $F \subset E$, $\tau^*(E)=\tau(\bar{E})$, where $\bar{E}$ is the closure of E. If $\tau_*(E)=\tau^*(E)$, the set E is called τ-measurable. The results of the preceding note are applied to the study of the connection between the capacity of the set of the points of convergence of a series of polynomials $\sum c_n p_n(z)$ (p_n a polynomial of degree n) and the quantity $\limsup |c_n|^{1/n}$. (1) If a sequence of polynomials $c_n p_n(z)$ converges on E, then there is a set $F_\tau \supset E$ such that $\limsup |c_n|^{1/n}=1/\tau_*(F_\tau)$. (2) No matter what F_τ we take, there is a sequence of polynomials $c_n p_n(z)$ converging in F_τ and such that $\limsup |c_n|^{1/n}=1/\tau_*(F_\tau)$. (3) Suppose that a sequence of polynomials $c_n p_n(z)$ converges on a bounded set E, that $\tau(E)=R$ and that $\limsup |c_n|^{1/n}=1/R$. Then the set of the points of convergence of $c_n p_n$ situated in that domain of the complement of E which contains the point can be enclosed in a finite or denumerable sum of closed sets each of capacity zero. *A. Zygmund* (Chicago, Ill.).

Picone, M. Teoria elementare della misura delle figure. Period. Mat. (4) **25**, 181–195 (1947).

Expository lecture.

Federer, Herbert. Dimension and measure. Trans. Amer. Math. Soc. **62**, 536–547 (1947).

A relation between topological dimension and Hausdorff measure was given by E. Szpilrajn [Fund. Math. **28**, 81–89 (1937)]: the dimension of a separable metric space X does not exceed m, $\dim X \leqq m$, if and only if a set Y of $(2m+1)$-space homeomorphic with X exists such that the $(m+1)$-dimensional Hausdorff measure of Y equals zero, $H^{m+1}_{m+1}(Y)=0$. The author [Trans. Amer. Math. Soc. **62**, 114–192 (1947); these Rev. **9**, 231] has studied a large class of k-dimensional (outer) measures over n-dimensional space, including the measures of Carathéodory, Hausdorff (H^k_n), Gross, Favard (integral geometric measure F^k_n) and another measure defined by the author. In the present paper he proves that the statement of Szpilrajn remains true if the Hausdorff measure is replaced by any of these measures and, more generally, by any measure ψ^k_n such that (i) if $H^k_n(A)=0$, then $\psi^k_n(A)=0$; (ii) if $\psi^k_n(A)=0$, then $F^k_n(A)=0$. A consequence of this result and of an earlier theorem of the author is the following statement. If A has a k-dimensional Hausdorff measure in n-space, if $B \subset A$ is the subset of points of A that are not (H^k_n, k)-restricted, then $\dim B \leqq k-1$. A formula is proved which expresses the k-dimensional Favard measure of a subset of n-space in terms of the m-dimensional Favard measures of its intersections with $(n-k+m)$-dimensional planes, as an integral over the set of all these planes.

L. Cesari (Bologna).

Giuliano, Landolino. Sulle trasformazioni assolutamente continue. Ann. Scuola Norm. Super Pisa (2) **12** (1943), 161–172 (1947).

The author considers a continuous mapping Φ: $x=x(u, v)$, $y=y(u, v)$, $(u, v) \varepsilon R$, where R is a simply connected, bounded Jordan region in the (u, v)-plane. Let R^0 be the interior of R. The author uses the concepts of bounded variation and absolute continuity, for such mappings Φ, in the sense of Cesari, and proves the following theorems. (1) Suppose that Φ is of bounded variation, and that for each set E of measure

zero in R^2 the image set $\Phi(E)$ is also of measure zero. Then Φ is absolutely continuous. An example is given to show that this condition is not necessary for Φ to be absolutely continuous. (2) If Φ is one-to-one, then it is absolutely continuous if and only if for each set E of measure zero in R^2 the image set $\Phi(E)$ is of measure zero. *T. Radó.*

Rosenthal, Arthur. What are set functions? Amer. Math. Monthly **55**, 14–20 (1948).

An expository article.

Nikodym, Otton Martin. Sur les êtres fonctionoïdes; une généralisation de la notion de fonction. C. R. Acad. Sci. Paris **226**, 375–377 (1948).

Nikodym, Otton Martin. Sur les êtres fonctionoïdes; procédé de complétion asymptotique. C. R. Acad. Sci. Paris **226**, 458–460 (1948).

Nikodym, Otton Martin. Échelle spectrale et intégration des êtres fonctionoïdes. C. R. Acad. Sci. Paris **226**, 541–543 (1948).

The author presents the foundations of a theory of "measurable functions" on Boolean algebras by axiomatizing the concept of a simple function (i.e., a finite linear combination of characteristic functions) and then going through a completion procedure, and discusses the relation of this theory to the one in which the analogue of the family of sets of the form $\{x: f(x)<\lambda\}$ is fundamental.

P. R. Halmos (Princeton, N. J.).

Kakutani, Shizuo. On equivalence of infinite product measures. Ann. of Math. (2) **49**, 214–224 (1948).

Two countably additive measures m and m' in an abstract set Ω, defined on the same Borel field, which satisfy $m(\Omega)=m'(\Omega)=1$, are called equivalent ($m\sim m'$) if each of them is absolutely continuous with respect to the other, and orthogonal ($m\perp m'$) if m' is singular with respect to m (then m is also singular with respect to m'). After a discussion of the Hellinger integral $\rho(m, m')=\int_\Omega\sqrt{m(d\omega)m'(d\omega)}$ the author considers the case where Ω is the direct product of a sequence of sets $\Omega_1, \Omega_2, \cdots$, and m and m' are product measures formed from measures m_n and m_n' in Ω_n such that $m_n(\Omega_n)=m_n'(\Omega_n)=1$, and proves the following theorem. If $m_n\sim m_n'$ for all n, then either $m\sim m'$ or $m\perp m'$, and the two cases occur according as the infinite product $\Pi=\prod_{n=1}^{\infty}\rho(m_n, m_n')$ is positive or zero. Moreover, $\rho(m, m')=\Pi$.

The proof is based on the fact that, if $\varphi_n(\omega_n)$ denotes the derivative of m_n' with respect to m_n, then the partial product $\prod_{n=1}^{k}\varphi(m_n, m_n')$ is the integral over Ω with respect to m of the function $\psi_k(\omega)=\prod_{n=1}^{k}\{\varphi_n(\omega_n)\}^{\frac{1}{2}}$. Now if $\Pi>0$ the sequence $\{\psi_k\}$ is easily seen to be convergent in square mean, and this implies that m' is absolutely continuous with respect to m and that $\varphi(m, m')=\Pi$. If $\Pi=0$ the sequence $\{\psi_k\}$ must converge in measure towards 0, and this easily implies that m' is singular with respect to m, and hence $\rho(m, m')=0$. *B. Jessen* (Copenhagen).

Kakutani, Shizuo. A proof of the uniqueness of Haar's measure. Ann. of Math. (2) **49**, 225–226 (1948).

A short proof based on the Fubini theorem. The first paragraph is ambiguous since Haar measure is defined by its "usual properties" without these properties having been explicitly stated. A more serious flaw is the implicit assumption, due to application of the Fubini theorem, that the "tribe" in $G\times G$ generated by compact sets of the form $A\times B$ is invariant under the transformation $(s, t)\to(s, st)$. The reviewer believes that this assumption can be justified, but not without considerable argument. *L. H. Loomis.*

Ridder, J. Über Stieltjessche Integrale und ihre Anwendung zur Darstellung linearer Funktionale. II. Nieuw Arch. Wiskunde (2) **22**, 220–240 (1948).

The Stieltjes integral definitions of part I [same Arch. (2) **22**, 171–188 (1946); these Rev. **8**, 451] are extended to functions of two variables, where in $\int f d\alpha$ the function $f(x, y)$ is assumed to have at most discontinuities of the first kind, i.e., for every (x, y) of the rectangle $a\leqq x\leqq b$, $c\leqq y\leqq d$ the limits $f(x+0, y)$, $f(x-0, y)$, $f(x, y+0)$, $f(x, y-0)$ and the double limits

$$f(x+0, y+0),\ f(x+0, y-0),\ f(x-0, y+0),\ f(x-0, y-0)$$

exist, while $\alpha(x, y)$ is of bounded variation in the sense that the interval function

$$\alpha(I\equiv(x_1, x_2; y_1, y_2))=\alpha(x_2, y_2)-\alpha(x_2, y_1)-\alpha(x_1, y_2)+\alpha(x_1, y_1)$$

is bounded. The principal result is that the most general linear continuous functional on the space of bounded functions having at most discontinuities of the first kind with least upper bound of $|f(x, y)|$ as norm is expressible in the form $\int f(x, y)d\varphi(E)$ where φ is a finitely additive bounded function on the additive class $\mathfrak{R}$ consisting of all open rectangles, all open segments parallel to the axes, and all points of the rectangle $(a, b; c, d)$. The integral $\int f d\varphi$ is the Moore-Smith limit of the finite approximating sums $\sum f(x_i, y_i)\varphi(E_i)$, (x_i, y_i) of E_i, in the sense of successive repartitions Π of $(a, b; c, d)$ into sets of $\mathfrak{R}$, and exists for all functions $f(x, y)$ of the space. This integral is expressible in terms of other integral definitions given in the paper.

T. H. Hildebrandt (Ann Arbor, Mich.).

Ridder, J. Zur Reduktion der n-fachen Integrale in abstrakten Räumen. Nieuw Arch. Wiskunde (2) **22**, 312–323 (1948).

The author treats his own definitions of abstract Riemann and Lebesgue integration [Fund. Math. **24**, 72–117 (1935)] and shows [extending his earlier result for $n=2$] that under suitable hypotheses an n-tuple integral may be evaluated by evaluating the corresponding iterated integrals. The complicated notation can be understood only by reference to the cited paper. The author does not say what advantages his presentation has over the usual discussion of Fubini's theorem. *P. R. Halmos* (Princeton, N. J.).

Theory of Functions of Complex Variables

Nehari, Zeev. A generalization of Schwarz' lemma. Duke Math. J. **14**, 1035–1049 (1947).

Schwarz's lemma is extended to nonuniform functions by the following theorem. Let $f(z)$ be a nonuniform function, regular for $|z|<1$ apart from a finite number of algebraic branch-points, and let $f'(z)$ be finite everywhere in $|z|<1$; let further, for all determinations of $f(z)$, $|f(z)|<1$ in $|z|<1$. Then $|f'(0)|\leqq 1$ for all possible values of $f'(0)$. Equality occurs only if $f(z)=\epsilon z$, where $|\epsilon|=1$. The importance of this result lies in the fact that it makes the powerful principle of subordination available in a greatly extended range of problems: if $f(z)$ is subordinate to $F(z)$, that is, if $f(z)=F(\omega(z))$ where $|\omega(z)|<1$ in $|z|<1$, then $\omega(z)$ need no longer be assumed as uniform. It suffices that it should

satisfy the conditions of the above theorem. The author gives several applications of which the following supplement of a classical theorem of Hurwitz may be mentioned. Let $f(z)=z+a_2z^2+\cdots$ be regular in $|z|<1$, and let $f(z)\neq 0$ for $z\neq 0$. Then every value w with $|w|<1/64$ is taken for some ζ in $|\zeta|<1$ so that $f'(\zeta)\neq 0$. The constant $1/64$ is best possible. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Ullrich, Egon. Konforme Abbildung eines Streifens mit Halbkreiskerbe. Z. Angew. Math. Mech. 25/27, 133–134 (1947).

The author comments on the conformal mapping of a parallel strip with a semicircular notch onto a half plane. A detailed account of his treatment of the problem will appear in a subsequent paper. *M. Heins.*

Greenstone, Leonard. Mapping by analytic functions. I. Conformal mapping of multiply-connected domains. Trans. Amer. Math. Soc. 63, 125–143 (1948).

The author uses the kernel function $K(z, \bar{t})$ of a multiply-connected domain D to construct the hyperbolic conformally invariant metric of the domain. He shows that if a sequence of domains B_n converges to B in the sense of Carathéodory, then the Gaussian curvatures $\mathfrak{J}_{B_n}$ of the corresponding metrics converge to $\mathfrak{J}_B$. He then obtains bounds for the distortion of the Euclidean length of an arc under conformal transformation, and also a generalisation of Poisson's formula to multiply-connected domains D in terms of $K(z, \bar{t})$. He also shows how many of his results can be extended to pseudo-conformal maps in space of more than two dimensions. *W. K. Hayman* (Exeter).

Schaeffer, A. C., and Spencer, D. C. A variational method in conformal mapping. Duke Math. J. 14, 949–966 (1947).

The theory of conformal mapping has in recent years obtained considerable progress by new and powerful methods [for a comprehensive account see G. M. Golusin, Uspekhi Matem. Nauk 6, 26–89 (1939); these Rev. 1, 49; English translation by T. C. Doyle, A. C. Schaeffer and D. C. Spencer, Office of Naval Research, Washington, 1947; these Rev. 8, 575]. Amongst these methods are variational arguments, usually applied to the boundary of the map by some extremal function of the problem in question. In the present paper the authors develop a method where variation is applied to some interior curve Γ of the map. It is not possible here to state the results in full: in short, when Γ is embedded in a field of neighbouring curves Γ_ν, having common end-points with Γ, then the first variation of conformally mapping functions $f_\nu(z)$ is determined.

W. W. Rogosinski (Newcastle-upon-Tyne).

Schiffer, Menahem. An application of orthonormal functions in the theory of conformal mapping. Amer. J. Math. 70, 147–156 (1948).

Let D be a domain in the z-plane, bounded by n smooth curves C_i and containing the point at infinity. Let Λ be the class of functions $f(z)$, which are regular in D, L^2 integrable over D, and possessing a uniform integral $F(z)$ in D. The kernel function $K(z, \bar{\xi})=\sum_{\nu=1}^{\infty} f_\nu(z)\overline{f_\nu(\xi)}$ of Λ with respect to D may be constructed from any complete orthonormal system $f_\nu(z)$ of Λ and is independent of the system chosen. The author extends these ideas and definitions to a wider class Ω, which may have a singularity at infinity. By constructing the kernel function $k(z, \bar{\xi})$ of Ω with respect to D in two ways he obtains a new proof of his inequality $S(D)\geqq 2\pi^{-1}E(D)$, where $S(D)$ is the span of the domain and $E(D)$ the area of the complementary domain, and he also obtains expressions for the functions which map D onto slit domains in terms of $K(z, \bar{\xi})$ and $k(z, \bar{\xi})$.

W. K. Hayman (Exeter).

Ríos, Sixto. Lectures on sequences of analytic functions and their applications. Revista Acad. Ci. Madrid 34, 291–316 (1940); 35, 5–31 (1941). (Spanish)

The classical theory of normal families of analytic functions of a complex variable, and kindred theories, are reviewed in a clear and orderly fashion. Criteria for normality are first developed. In particular, the fundamental criterion of Montel is obtained from Bloch's theorem. Next, criteria for the uniform convergence of sequences of analytic functions are discussed. The following result of Landau and Carathéodory is given. If a sequence of functions analytic in a domain D, such that no member of the family takes on either of the finite values a, b, converges on a set of points having an interior accumulation point, then the sequence converges uniformly in the interior of D. It is observed that this result chronologically precedes the Montel criterion for normality. Applications are made to the Picard theorems, the theorem of Schottky, and so on. Irregular points, quasi-normal families, lines of Julia and families of meromorphic functions are discussed, and fundamental theorems of conformal mapping are developed. There is a rather extensive bibliography. *E. F. Beckenbach.*

Combes, Jean. Familles normales sur une surface de Riemann. C. R. Acad. Sci. Paris 226, 379–381 (1948).

Let F be the Riemann surface, of m finite sheets, of the function $[E(z)]^{1/m}$, where $E(z)$ is an entire function with zeros at $z_1, z_2, \cdots$, with $|a_i|\leqq|a_{i+1}|$. Let $\bar{D}$ be a domain in the z-plane, and let D be the (m-sheeted) part of F which projects on $\bar{D}$. The author studies families of single-valued analytic functions on D and shows, in particular, that in order that a family be normal in D it is necessary and sufficient that it be normal at each point of D, and that if each member of a family omits two distinct finite values a and b then the family is normal. *E. F. Beckenbach.*

Heins, Maurice. Entire functions with bounded minimum modulus; subharmonic function analogues. Ann. of Math. (2) 49, 200–213 (1948).

Soit $m(r)$ le minimum et $M(r)$ le maximum du module d'une fonction entière $f(z)$ pour $|z|=r$, $n(r)$ le nombre des zéros de $f(z)$ appartenant au cercle $|z|\leqq r$. L'auteur donne de nouveaux résultats en relation avec le théorème de Wiman d'après lequel, si $m(r)=O(1)$, on a $\alpha=\liminf_{r\to\infty} r^{-1}\log M(r)>0$. Il montre que, dans ces conditions, si α est fini, $r^{-1}\log M(r)$ tend vers α, d'où il résulte, d'après un théorème connu [que l'auteur attribue à Titchmarsh, mais qui d'après Titchmarsh même, se trouve déjà dans Valiron, Ann. Fac. Sci. Univ. Toulouse (3) 5, 117–257 (1914), en particulier, pp. 241–247], $n(r)\sim\pi^{-1}\alpha r^1$. Le grand intérêt du mémoire, en dehors des résultats indiqués, est dans l'emploi de la théorie des fonctions sous-harmoniques. Le théorème sur la représentation des fonctions entières d'ordre inférieur à un, à zéros alignés, par une intégrale portant sur le produit de $n(x)$ par une fonction convenable de x et r est remplacé par un théorème sur la représentation d'une fonction sous-harmonique d'ordre inférieur à un par une distribution de masses; le théorème de Milloux-Schmidt est formulé pour les fonctions sous-harmoniques ainsi que l'un des théorèmes utilisés par l'auteur dans son mémoire sur le principe de

Lindelöf-Phragmén [Trans. Amer. Math. Soc. 60, 238–244 (1946); ces Rev. 8, 371]. Dans ce même ordre d'idées, l'auteur donne une nouvelle démonstration du théorème attribué à Titchmarsh. *G. Valiron* (Paris).

Valiron, Georges. Remarques sur les domaines d'univalence des fonctions entières d'ordre inférieur à ½. Bull. Sci. Math. (2) 71, 25–32 (1947).

If $f(z)$ is an integral function of order less than ½ and R is arbitrarily large there exists a region of the z-plane conformally represented by $Z=f(z)$ on $|Z|<R$ possibly cut radially to a point on the circumference. If the order of magnitude of $f(z)$ does not exceed $\exp\{(\log|z|)^2\}$ this region will subtend an arbitrarily small angle at the origin if R is sufficiently large. This second result was given earlier but with the possibility of two radial cuts [Valiron, same Bull. (2) 59, 298–320 (1935), p. 314] and is now extended to the class of functions of zero order satisfying the "regularity condition" $r\int_r^\infty x^{-2}n(x)dx=o\{\int_1^r x^{-1}n(x)dx\}$ ($n(r)$ is the number of zeros of $f(z)$ in $|z|<r$). Related regularity conditions bearing on the power series of $f(z)$ and the distribution of the zeros are considered and the theory extended to functions analytic and uniform for $|z|>\rho$ and to certain classes of periodic functions. The new basis of the argument is the fact that if $|f(z)|$ is constant and $f'(z)\neq 0$ on the boundary of a simply connected domain Δ then the number of zeros of $f(z)$ in Δ exceeds the number of zeros of $f'(z)$ in Δ by unity. One at least of the circular sheets of the Riemann surface corresponding to Δ can have no more than a single winding point. The results relating to the size of the regions are derived by comparison with theorems of Valiron and of Littlewood on integral functions of zero order [cf. Valiron, Lectures on the General Theory of Integral Functions, Toulouse, 1923, pp. 132–137]. *A. J. Macintyre.*

Valiron, Georges. Le théorème et le programme de Borel. Revista Unión Mat. Argentina 12, 297–309 (1947). (Spanish)

Valiron, Georges. Chemins de détermination. Fonctions inverses. Domaines d'univalence. Revista Unión Mat. Argentina 12, 310–325 (1947).

Lectures at the University of Buenos Aires.

Shah, S. M. A note on the derivatives of integral functions. Bull. Amer. Math. Soc. 53, 1156–1163 (1947).

L'auteur démontre deux théorèmes et en donne des applications. Soit $f(z)$ une fonction entière; $M(r)$ désigne le maximum de son module pour $|z|=r$, $M'(r)$ la fonction analogue pour $f'(z)$, $\nu(r)$ le rang du terme maximum du développement de Taylor de $f(z)$. (I) Pour r infini, on a

$$\liminf \frac{\log\{M'(r)/M(r)\}}{\log r}=\lambda-1,$$

λ étant l'ordre inférieur de $f(z)$. (II) Pour r infini, on a

$$\liminf M'(r)/M(r)\leqq\liminf \nu(r)/r\leqq\limsup \nu(r)/r\leqq\limsup M'(r)/M(r),$$

et des inégalités analogues où $M(r)$ et $M'(r)$ sont remplacés par les fonctions relatives aux dérivées d'ordre s et $s+1$. Les démonstrations utilisent des résultats antérieurs de l'auteur [J. Univ. Bombay (N.S.) 13, part 3, 1–3 (1944); Math. Student 10, 80–82 (1942); ces Rev. 6, 206; 4, 137]. Le premier théorème avait été donné par S. K. Bose avec une démonstration insuffisante [J. Indian Math. Soc. (N.S.) 10, 77–80 (1946); ces Rev. 9, 276]. *G. Valiron.*

v. Szökefalvi Nagy, Gyula (Julius). Über einen Satz von J. Dieudonné. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 60, 700–716 (1941). (Hungarian. German summary)

J. Dieudonné [Ann. of Math. (2) 31, 79–116 (1930)] proved the following analogue to Rolle's theorem. Let a and b be two successive zeros of a real entire function of genus 0 or 1 and with only real zeros. Then exactly one zero of the derivative of the function $g(z)=e^{\pm iq/R}f(z)$ lies in the circle $|z-(a+b)/2|\leqq R$. Nagy now shows that under the same hypotheses exactly one zero of the derivative of the function $G(z)=e^{qi}g(z)$, q an arbitrary real number, lies in a region bounded by the line-segment (a, b) and by a curve which may be taken as a certain arc either of a circle with a radius of R and passing through a and b or of the bicircular quartic $(y/|z-a|^2)+(y/|z-b|^2)=1/R$. *M. Marden* (Milwaukee, Wis.).

Gel'fond, A. O., and Ibragimov, I. I. On functions whose derivatives are zero at two points. Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 547–560 (1947). (Russian)

The authors discuss several cases in which $f^{(\nu_s)}(1)=0$ for a sequence $\{\nu_s\}$ and $f^{(k)}(0)=0$ for $k\neq\nu_s$ imply that $f(z)\equiv 0$. Let

$$\sigma_n^{\nu_n}=\frac{\nu_n!}{(\nu_n-\nu_{n-1})!\cdots(\nu_1-\nu_0)!\nu_0!}.$$

Then $f(z)\equiv 0$ in the following cases: (A) $f(z)$ is regular in $|z|\leqq R$, where $R>\sigma=\liminf\sigma_n$; (B) $f(z)=\sum_0^\infty a_k z^{\nu_k}$ is an entire function, $\lim\sigma_n=\infty$ and $\lambda_n=\min_{k\geq n}\sigma_k$, $|a_k|<(\delta_k/\nu_k)\lambda_k^{-\nu_k}$ and $\sum\delta_k<\infty$; (C) $f(z)$ is an entire function of finite order $\rho>1$ and $\lim\sigma_n/\log\nu_n=1/\rho$.

If $\{\nu_s\}$ is an arithmetic progression, $\nu_s=ps-1$, $f(z)\equiv 0$ if it is entire and of exponential type $a<p/e$; moreover, p/e cannot be replaced by a smaller number for sufficiently large p. *R. P. Boas, Jr.* (Providence, R. I.).

Wille, R. J. An outer limit of nonconformalness, for which Picard's theorem still holds. Nederl. Akad. Wetensch., Proc. 50, 904–908 = Indagationes Math. 9, 415–419 (1947).

For the validity of Picard's theorem in the case of quasiconformal mappings it is necessary and sufficient that the maximum dilation $D(r)$ on $|z|=r$ be such that $\int^\infty D(r)^{-1}r^{-1}dr$ diverges. This result is known and a more direct proof can be given [O. Teichmüller, Deutsche Math. 3, 621–678 (1938); H. Grötzsch, Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. 80, 503–507 (1928)]. *L. Ahlfors.*

Milloux, Henri. Les dérivées des fonctions méromorphes et la théorie des défauts. Ann. Sci. École Norm. Sup. (3) 63, 289–316 (1947).

Der Verf. beweist in Verallgemeinerung seiner frühern Ergebnisse die folgende fundamentale Ungleichung. Es sei $f(z)$ eine meromorphe Funktion und $a_1\neq a_2\neq\cdots\neq a_p$, sowie $b_1\neq b_2\neq\cdots\neq b_q$. Dann gilt

$$pqT(r,f)<\bar{N}(r,f)+q\sum_1^p N\left(r,\frac{1}{f-a_i}\right)+\sum_{i=1}^q N\left(r,\frac{1}{f'-b_i}\right)+S'(r),$$

wobei die Funktionen T, N, $\bar{N}$ im Sinne von R. Nevanlinna definiert sind und $S'(r)$ gegenüber $T(r,f)$ von geringerer Grössenordnung ist. Auf Grund dieser Ungleichung lässt sich die Nevanlinna'sche Defekttheorie auf den Fall verall-

gemeinern, dass neben Werten der Funktion auch solche der ersten Ableitung betrachtet werden. *W. Saxer.*

Baganas, Nicolas. Sur les valeurs algébriques d'une fonction algébroïde. C. R. Acad. Sci. Paris **226**, 545–546 (1948).

Announcement of results concerning algebraic values of algebroid functions which are related to earlier studies of Montel [J. Math. Pures Appl. (9) **20**, 305–324 (1941); these Rev. **4**, 7]. *M. Heins* (Providence, R. I.).

Martis in Biddau, Silvia. I funzionali lineari algebrici e abeliani. Rend. Circ. Mat. Palermo **62**, 289–358 (1941).

Das Ziel der Arbeit ist die linearen, lokal-analytischen Funktionale zu studieren, deren Indikatrix eine algebraische Funktion ist oder durch ein abelsches Integral gegeben wird. Nach einleitenden Definitionen werden im ersten Kapitel die Funktionale untersucht, deren Indikatrix ein bestimmter Zweig u_r einer algebraischen Funktion u ist, und die in Bezug auf ein gewisses Schnittsystem definiert sind. Es zeigt sich, dass die Summe der zu den verschiedenen Zweigen einer algebraischen Funktion gehörenden Funktionale ein elementares Funktional bildet, während ein einzelner Zweig selbst ein Funktional definiert, das sich im allgemeinen aus einem elementaren Funktional und einer Summe von abelschen Funktionalen über die verschiedenen Schnitte zusammensetzt. Letztere werden durch ein (offenes oder geschlossenes) Kurvenintegral auf der Riemann'schen Fläche der algebraischen Funktion u definiert:

$$A[y(t)]=(2\pi i)^{-1}\int_C u(t)y(t)dt.$$

Nach einigen bekannten Betrachtungen über den Hauptteil (partie finie) uneigentlicher Integrale werden die in einem algebraischen Funktional auftretenden, in Bezug auf einen Zweig u_r definierten, abelschen Funktionale durch ein Integral längs den beiden Rändern des Schnittes oder dessen Hauptteil, wenn das Integral uneigentlich wird, und ein elementares Funktional dargestellt.

Im zweiten Kapitel wird der Begriff der abelschen Funktionale so verallgemeinert, dass ein gegebenes abelsches Funktional bei birationalen Transformationen invariant bleibt, wenn Variablen und Integrationsweg gleich transformiert werden; dabei muss der Integrationsweg gewisse Bedingungen erfüllen, die in der Folge abgeschwächt werden und für Zykeln ganz wegfallen. Mit Hilfe der birationalen Invarianz wird eine Normalform für abelsche Funktionale definiert. Nun wird nochmals die Zerlegung eines algebraischen Funktionals in ein elementares und ein einziges abelsches Funktional betrachtet, wobei letzteres jetzt in Bezug auf einen nullhomologen Zykel definiert ist, der auf der Riemann'schen Fläche der algebraischen Indikatrix ein zusammenhängendes, einfach bedecktes Gebiet beranden muss. Umgekehrt erweist sich jedes abelsches Funktional in Bezug auf einen solchen Zykel als ein algebraisches Funktional. Nun wird entsprechend der Vielfalt der lokal-analytischen Funktionen ein algebraisches Funktional im weitern Sinne definiert, dessen Indikatrix in jedem zusammenhängenden Teil ihres Definitionsbereiches eine algebraische Funktion ist mit der Bedingung, dass die Gesamtanzahl aller Pole endlich bleibt. Auch hier wird wiederum die Zerlegung in ein elementares Funktional und eine eventuell unendliche Summe von abelschen Funktionalen relativ nullhomologen Zykeln der jeweils zugehörigen algebraischen Funktionen durchgeführt, wobei jedoch immer nur endlich viele einen von Null verschiedenen Beitrag geben. Im weitern wird nun gezeigt, dass ein abelsches Funktional in Bezug auf einen beliebigen nullhomologen Zykel immer ein verallgemeinertes algebraisches Funktional ist, und abschliessend, dass ein abelsches Funktional auf einer Riemann'schen Fläche vom Geschlechte $p\geqq 1$ in Bezug auf einen beliebigen Zykel bis auf ein verallgemeinertes algebraisches Funktional immer durch eine Linearkombination mit ganzen Koeffizienten von $2p$ abelschen Funktionalen in Bezug auf die Elemente einer Zyklenbasis dargestellt wird. *H. G. Haefeli* (Cambridge, Mass.).

***Keinänen, Viljo. Über die vermischte Randwertaufgabe der Halbebene bei unendlich vielen Randintervallen.** Akademische Abhandlung, Helsinki, 1947. 80 pp.

This memoir is concerned with the theory of Abelian integrals on transcendental two-sheeted Riemann surfaces with real ramification points and as such constitutes a continuation of earlier investigations of Myrberg [Acta Math. **76**, 185–224 (1945); Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. nos. **14** (1943), **31** (1945); C. R. Dixième Congrès Math. Scandinaves 1946, pp. 77–96; these Rev. **7**, 57, 428; **8**, 509]. The author is concerned with the two-sheeted Riemann surface over the y-plane defined by $y^2=g(x)$, where g ($\not\equiv 0$) is entire in x, is real for x real, and has infinitely many zeros which are all real and simple. It is assumed that there are both infinitely many positive and infinitely many negative zeros. In the earlier work of Myrberg, attention was confined to the case where all but a finite number of zeros were positive. The author investigates the properties of the elementary normal integrals of the three kinds; an arbitrary location on the surface is admitted for the singularity. In addition, the author treats mixed boundary value problems for harmonic functions in a half-plane. A special case of the situation treated by the author was studied by Hornich [Monatsh. Math. Phys. **40**, 241–282 (1933)]. *M. Heins* (Providence, R. I.).

Hervé, Michel. Sur les fonctions automorphes de n variables complexes. C. R. Acad. Sci. Paris **226**, 462–464 (1948).

Let D be a bounded domain in the space of n complex variables $x=(x_1, \cdots, x_n)$ and γ a group of analytic automorphisms $y=f(x)$ of D. Suppose that γ is discontinuous on D and that there exists a fundamental region which is compact with respect to D. Denote by dy/dx the Jacobian of the transformation $y=f(x)$. A function $\theta(x)$ is called a Fuchsian function of dimension k relative to γ if it is analytic throughout D and satisfies there the equation $\theta(y)(dy/dx)^k=\theta(x)$ for all transformations in γ. The author indicates a proof that the number of linearly independent $\theta(x)$, for any fixed $k>0$, is less than Ak^n, where A is a positive constant depending only upon D and γ. This is a generalization of a theorem proved by Hua [Ann. of Math. (2) **47**, 167–191 (1946); these Rev. **7**, 429]. *C. L. Siegel* (Princeton, N. J.).

Lelong, Pierre. Sur les séries de Taylor à deux variables, à coefficients entiers. C. R. Acad. Sci. Paris **226**, 210–212 (1948).

Let $F(x, y)=\sum a_{mn}x^m y^n$, where the coefficients a_{mn} are integers, be analytic and single-valued in the domain $[|x|<1, |y|<1]$ except on certain analytic manifolds. The function $F(x, y)$ may be transcendental even if it is analytic in a point of the boundary of the domain, but the author

proves that $F(x, y)$ is a rational function if it is analytic in a point of the manifold $[|x|=1, |y|=1]$. The theorem is proved by an investigation of the determinants $\|A_{i+j}(\xi)\|$ formed with the coefficients $A_n(\xi)$ in the development $F(\xi^{-1}, \eta^{-1})=\sum_{n=0}^{\infty}A_n(\xi)\eta^{-n}$. References are given to analogous theorems concerning functions of one variable.

H. Tornehave (St. Johns, Que.).

Kriszten, Adolf. Areolar monogene und polyanalytische Funktionen. Comment. Math. Helv. **21**, 73–78 (1948).

Ist A der Operator $\partial/\partial x+i\partial/\partial y$, so sind die analytischen Funktionen f der komplexen Variabeln $z=x+iy$ durch die Gleichung $Af=0$ definiert. Pompeiu und andere haben die Funktionen betrachtet, die der Gleichung $A^2f=0$ genügen und sie areolar monogen genannt. Der Verfasser schlägt für diese Funktionen den Namen "bianalytisch" (als Spezialfall von "polyanalytisch" (s.u.)) vor und zeigt: (1) ist $f=u+iv$, $A^2f=0$, so ist $\Delta^2u=\Delta^2v=0$; (2) ist $\Delta^2u=0$, so existiert v, sodass $A^2(u+iv)=0$; (3) sind f und g in einem Gebiete und auf seinem Rande s bianalytisch, so gilt $\int_s(gAf-fAg)dz=\sigma$ (erster Integralsatz); (4) ist f in einem einfach zusammenhängenden Gebiet und auf seinem Randes bianalytisch, so gilt für z in diesem Gebiet

$$2\pi i f(z)=\int_s\left\{\frac{f(\zeta)}{\zeta-z}-\frac{\overline{\zeta-z}}{2(\zeta-z)}Af(\zeta)\right\}d\zeta$$

(zweiter Integralsatz). Sodann betrachtet der Verfasser die komplexen Funktionen, welche der Gleichung $A^nf=0$ genügen und nennt sie "polyanalytisch." Ihre Komponenten sind polyharmonisch und es gelten ein erster und ein zweiter Integralsatz in entsprechend verallgemeinerter Form.

W. Nef (Fribourg).

Mellihzon, A. S. On a question about Galois complexes. Doklady Akad. Nauk SSSR (N.S.) **58**, 981–984 (1947). (Russian)

Let j satisfy an algebraic equation of the form

$$j^n+P_1j^{n-1}+\cdots+P_n=0;$$

let ζ be a generalized complex variable,

$$\zeta=x_0+jx_1+\cdots+j^{n-1}x_{n-1},$$

where the x_k are independent real variables, and let $F(\zeta)$ be a generalized complex function of ζ,

$$F(\zeta)=u_0(\zeta)+ju_1(\zeta)+\cdots+j^{n-1}u_{n-1}(\zeta),$$

where the $u_k(\zeta)$ are real functions of the n real variables $x_0, \cdots, x_{n-1}$. The author obtains linear partial differential equations, analogous to the Cauchy-Riemann equations, which necessarily are satisfied if $dF/d\zeta$ exists. Analogues of the Cauchy integral theorem and the theorem of Morera are established for the case $n=3$. *E. F. Beckenbach.*

Loonstra, F. Sur les intégrales multiples dans les corps évalués et algébriquement-fermés. Nederl. Akad. Wetensch., Proc. **50**, 1273–1284 = Indagationes Math. **9**, 599–610 (1947).

Let T be an algebraically closed field complete with respect to a non-Archimedean valuation. Generalizing a definition given by Schnirelmann [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1938**, 487–498] for functions of one variable, the author defines an integral for functions $f(x_1, \cdots, x_s)$ of s variables in T as follows. For each fixed $k=1, \cdots, s$, let $g_{ki}(x)=x^{n_{ki}}+c_{ki}x^{m_{ki}}+\cdots$ be a sequence of polynomials over T such that $n_{ki}>m_{ki}>\cdots$, $\lim_{i\to\infty}(n_{ki}-m_{ki})=\infty$; let the roots of $g_{ki}(x)$ be denoted by $a_{ki\,1}, a_{ki\,2}, \cdots$, and assume they are distinct and of absolute value 1. If $\alpha_1, \cdots, \alpha_s; \rho_1, \cdots, \rho_s$ are arbitrary elements of T, the s-fold integral of $f(x_1, \cdots, x_s)$ over the "circles" with centers $\alpha_1, \cdots, \alpha_s$ and radii $\rho_1, \cdots, \rho_s$ is defined as

$$\lim_{i\to\infty}\frac{1}{n_{1i}\cdots n_{si}}\sum f(\alpha_1+\rho_1a_{1ih_1}, \cdots, \alpha_s+\rho_sa_{sih_s})$$

(the summation being over $1\leqq h_1\leqq n_{1i}, \cdots, 1\leqq h_s\leqq n_{si}$), provided this limit exists and is independent of the $g_{ki}(x)$. For functions which can be expanded in power series this integral always exists and is equal to $f(\alpha_1, \cdots, \alpha_s)$. There are a number of other results, leading up to integral representations, analogous to the Cauchy formulas, for analytic functions and their derivatives. From this are derived the familiar (in the classical case) Cauchy inequalities for the derivatives of an analytic function. *I. S. Cohen.*

Theory of Series

Copsey, E. H., Frazer, H., and Sawyer, W. W. Empirical data on Hilbert's inequality. Nature **161**, 361 (1948).

Frazer [J. London Math. Soc. **21**, 7–9 (1946); these Rev. **8**, 259] improved Hilbert's inequality to

$$\sum_{r=1}^{N}\sum_{s=1}^{N}\frac{a_ra_s}{r+s-1}\leqq\lambda_N\sum_{r=1}^{N}a_r^2,$$

with $\lambda_N=N\sin(\pi/N)$, $N\geqq 2$. The authors present the results of numerical computations of λ_N for $N=2, 3, 4, 5, 10, 20$, showing that Frazer's value for λ_N is not the best possible. *R. P. Boas, Jr.* (Providence, R. I.).

Birindelli, Carlo. Relazioni ricorrenti tra particolari procedimenti (f, g) di Gronwall. Estensione dei metodi (f, g) per la sommazione generalizzata delle serie multiple. Rend. Circ. Mat. Palermo **63**, 1–32 (1942).

Two functions $f(w)$ and $g(w)=\sum b_nw^n$ satisfying appropriate conditions determine a Gronwall [Ann. of Math. (2) **33**, 101–117 (1932)] method (f, g) of summability by which a series $\sum u_n$ is summable to s if the identity

$$g(w)\sum_{n=0}^{\infty}u_n[f(w)]^n=\sum_{n=0}^{\infty}b_nU_nw^n$$

determines a sequence $U_0, U_1, U_2, \cdots$ convergent to s. Continuing his studies [Ann. Scuola Norm. Super. Pisa (2) **8**, 241–270 (1939); these Rev. **2**, 90] of these methods the author considers the method

$$Q_k\equiv(1-(1-w)^{2^{-k}}, (1-w)^{-2^{-k}}), \qquad k=1, 2, 3, \cdots,$$

giving a simple formula expressing the elements of the Q_{k+1} transform of $\sum u_n$ in terms of the elements of the Q_k transform of $\sum u_n$. The method Q_1 was defined and studied, without use of the Gronwall theory, by Obrechkoff [C. R. Acad. Sci. Paris **182**, 307–309 (1926)].

The Gronwall theory of the Vallée-Poussin methods V_k of order k $(k=0, 1, 2, \cdots)$ is discussed. It is erroneously asserted that V_0 includes C_r (Cesàro) for each $r>0$. The transformation V_0, written as a sequence-to-sequence transformation, is $V_n^{(0)}=2^{-n}\sum_{k=0}^{n}\binom{n}{k}s_k$. This is identical with the classic Euler transformation E_r, defined by

$$\sigma_n^{(r)}=\sum_{k=0}^{n}\binom{n}{k}r^k(1-r)^{n-k}s_k,$$

for the case $r=\frac{1}{2}$. Thus the assertion is contradicted by the well-known fact that if $r, s>0$, then neither of E_r and C_s includes the other. [See W. A. Hurwitz, Bull. Amer. Math. Soc. 28, 17–36 (1922), p. 27.] Birindelli's erroneous supporting argument is based on the Silverman-Toeplitz theorem giving necessary and sufficient conditions that the sequence-to-sequence transformation $\sigma_n=\sum_{k=0}^{n}a_{nk}s_k$ be regular, i.e., such that $\lim \sigma_n=\lim s_n$ whenever the sequence s_n is convergent. The author's statement of this theorem is faulty. In the first place, he calls the transformation regular if it is regular in the above sense and has several other additional properties (involving infinite limits, removal of elements from sequences, Cauchy products, etc.) which many regular transformations fail to have; such definitions of regularity were sometimes used in France many years ago. In the second place, the familiar norm condition $\sum_{k=0}^{n}|a_{nk}|<M$, $n=0, 1, 2, \cdots$, is miswritten in the form "$\sum_{k=0}^{\infty}|a_{nk}|$ convergente."

The second part of the paper treats Gronwall summability of double series, the transformation involved being the series-to-sequence transformation

$$g(w_1)g(w_2)\sum_{j,k=0}^{\infty}u_{jk}[f(w_1)]^j[f(w_2)]^k=\sum_{m,n=0}^{\infty}b_mb_nU_{mn}w_1^mw_2^n$$

or a related sequence-to-sequence transformation. Five examples are given and the methods are applied to double Fourier series.

The third part of the paper gives extensions to double series of Gronwall's fundamental theorems on his methods. For a more recent and more extensive treatment of these extensions see Birindelli [Portugaliae Math. 6, 1–32 (1947); these Rev. 9, 27]. *R. P. Agnew* (Ithaca, N. Y.).

Cesari, Lamberto. Sulla moltiplicazione delle serie doppie. Ann. Scuola Norm. Super. Pisa (2) 12 (1943), 189–204 (1947).

Proofs of three theorems, previously announced [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 289–292 (1946); these Rev. 8, 260], on convergence and restricted Cesàro summability of Cauchy products of double series. The Cauchy product of $\sum a_{mn}$ and $\sum b_{mn}$ is the series $\sum c_{mn}$, where

$$c_{mn}=\sum_{r+u=m}\ \sum_{s+v=n}a_{rs}b_{uv}.$$

[This formula is misprinted with ∞ at the top of each $\sum$.] Theorem 1 says that if $\sum a_{mn}$ and $\sum b_{mn}$ converge to A and B and $\lim_{m+n\to\infty}a_{mn}=0$, $\lim_{m+n\to\infty}b_{mn}=0$ then $\sum c_{mn}$ is restrictedly summable $(C, 1, 1)$ to AB. Theorem 2 says that if $\sum a_{mn}$ and $\sum b_{mn}$ converge restrictedly to A and B, and if there exist constants $P, Q, R, \sigma\ (<1)$, and $\tau(<1)$ such that their partial sums s_{mn} and s'_{mn} are dominated by

$$P+Q[(m+1)/(n+1)]^{\tau}+R[(n+1)/(m+1)]^{\tau},$$

then $\sum c_{mn}$ is restrictedly summable $(C, 1, 1)$ to AB. Theorem 3 generalizes a classic theorem of Mertens [J. Reine Angew. Math. 79, 182–184 (1874)] on simple series. If $\sum a_{mn}$ and $\sum b_{mn}$ converge to A and B, if $\sum a_{mn}$ converges by rows and by columns, and if $\sum b_{mn}$ converges absolutely, then the Cauchy product series $\sum c_{mn}$ converges to AB, converges by rows to AB, and converges by columns to AB. *R. P. Agnew* (Ithaca, N. Y.).

Wintner, Aurel. On Riemann's reduction of Dirichlet series to power series. Amer. J. Math. 69, 769–789 (1947).

The author attaches a number of observations of a Tauberian nature to the classical formula

$$\Gamma(s)f(s)=\int_0^{\infty}x^{s-1}F(e^{-x})dx,$$

where $f(s)=\sum_1^{\infty}a_nn^{-s}$, $F(r)=\sum_1^{\infty}a_nr^n$. If the Dirichlet series converges for $\sigma>1$ and if the coefficients $\{a_n\}$ have an Abel mean so that (1) $\lim_{r\to1}(1-r)F(r)=l$, then they also have a Dirichlet mean (2) $\lim_{\epsilon\to0}\epsilon f(1+\epsilon)=l$. Since (2) does not imply (1), the author raises the question of when (1) may be concluded from properties of $f(s)$ and $F(r)$. The main result is the following. Suppose that (i) the Dirichlet series converges for $\sigma>1$ and there exists an l such that $f^*(s)=f(s)-l\zeta(s)$ has boundary values on $\sigma=1$ in the sense that $f^*(\sigma+it)$, when $\sigma\to1$, converges in the mean of order one on each finite interval $-T<t<T$, and (ii) $F(r)$ is a monotone function of r as $r\to1$. Then (1) holds. Here (ii) may be replaced by conditions of the form $\sum_{n<x}a_n=O(x)$ or $\sum_{n<x}na_n=O_L(x^2)$, while (i) is implied by integrability conditions on $F^*(r)=F(r)-lr(1-r)^{-1}$ of the form $\int_0^1(1-r)^{p-1}|F^*(r)|^pdr<\infty$ for some $p\geqq1$. *E. Hille.*

Minakshisundaram, S., and Rajagopal, C. T. Postscript to a Tauberian theorem. Quart. J. Math., Oxford Ser. 18, 193–196 (1947).

The authors prove a Tauberian theorem for Rieszian means which is a one-sided analogue of a theorem (theorem A′) given by them in an earlier paper [same J. 17, 153–161 (1946); these Rev. 8, 147]. The full result is too complicated to set down here but is of a similar nature to their earlier theorem. The chief difference is that the earlier condition on $|a_n|$ (a_n is the general term of the series) is replaced by two alternative conditions; one of these is a one-sided condition on a_n and the other is independent of a_n. The connexion between the result and related theorems due to other authors is described. *R. A. Rankin.*

de Marchin, R. Note sur les fractions continues. Nieuw Tijdschr. Wiskunde 35, 196–201 (1947).

Let r_1 be the root of $x^2+px+q=0$ $(p^2-4q>0)$ which is the smaller in absolute value. The author proves that $-r_1=K[-q/-p]$. The result is known [Perron, Die Lehre von den Kettenbrüchen, Teubner, Leipzig, 1929, p. 276], since $r_1+r_2=-p$. *W. Leighton* (St. Louis, Mo.).

Reed, I. S. On a continued fraction related to Euler's constant. Revista Ci., Lima 49, 239–245 (1947).

Fourier Series and Generalizations, Integral Transforms

Boas, R. P., Jr. Inequalities for the coefficients of trigonometric polynomials. II. Nederl. Akad. Wetensch., Proc. 50, 759–762 = Indagationes Math. 9, 369–372 (1947).

Let $F(t)=\sum_{-n}^{n}a_ke^{ikt}$ be a real trigonometric polynomial, γ any positive number and k any integer exceeding $\frac{1}{2}n$; then the author proves that

$$|a_0|+2\gamma|a_k|\leqq C_{\gamma}\int_0^{2\pi}|F(t)|dt,$$

where $C_\gamma = 1/(2\pi - 4\delta)$, and δ is the smallest positive root of $\sin\delta = \frac{1}{2}\gamma(\pi - 2\delta)$. The constant C_γ is the best possible. This is an extension and improvement on results previously given by van der Corput and Visser [same Proc. 49, 383–392 = Indagationes Math. 8, 238–247 (1946); these Rev. 8, 148] and the author [same Proc. 50, 492–495 = Indagationes Math. 9, 298–301 (1947); these Rev. 9, 88]. Some more inequalities are given, in particular

$$|a_k| \leqq \tfrac{1}{2}\cos(\pi/(p+2))\int_0^{2\pi} |F(t)|\,dt,$$

where p is the largest odd integer with $pk \leqq n$.

A. C. Offord (Newcastle-upon-Tyne).

Gagaev, B. M. On the convergence of trigonometric series. Doklady Akad. Nauk SSSR (N.S.) **59**, 5–8 (1948). (Russian)

Let $f(t)$ be of period 2π, L-integrable, even, and let $f(t) \sim \sum a_n \cos nt$. Hardy and Littlewood [J. London Math. Soc. 7, 252–256 (1932)] proved that, if $f(t) = o(\log 1/t)^{-1}$ as $t \to 0$ and if $a_n = O(n^{-\delta})$ for some $\delta > 0$, then $\sum a_n = 0$. The author proves the following extension. Suppose that $f(t) = o(\varphi(t))$ as $t \to +0$ and that $a_n = O(\psi(n^{-1}))$, where $\varphi(\alpha)$ and $\psi(\alpha)$ are two positive and monotone functions tending to 0 with α and such that

$$\int_{1/n}^{\omega(n)} t^{-1}\varphi(t)\,dt < A, \quad [\omega(n)]^{-1}\sum_{k=1}^{n}{}'\psi(1/k)/|n-k| \to 0$$

as $n \to \infty$. Here A is independent of n, the prime signifies that $k \neq n$ in the summation and $\omega(n)$ decreases monotonically to 0 as $n \to +\infty$ in such a way that $n\omega(n) \to \infty$. Under these conditions, $\sum a_n = 0$. For similar extensions of the Hardy-Littlewood result, see Morgan [Ann. Scuola Norm. Super. Pisa (2) 4, 373–382 (1935)]. *A. Zygmund.*

Korevaar, J. A characterization of the sub-manifold of $C[a, b]$ spanned by the sequence $\{x_k^n\}$. Nederl. Akad. Wetensch., Proc. 50, 750–758 = Indagationes Math. 9, 360–368 (1947).

Suppose that $\sum n_k^{-1} < \infty$, where n_k is a monotone increasing sequence of integers. The central result is that the linear extension of $\{x^{n_k}\}$ in $C(0, 1)$ is the set of all power series $\sum a_k x^{n_k}$ converging for $0 \leqq x < 1$ for which the limit as $x \to 1$ exists. This extends the result from the case $n_{k+1}/n_k > c > 1$ established by Clarkson and Erdös [Duke Math. J. 10, 5–11 (1943); these Rev. 4, 196]. The methods are those of Clarkson and Erdös. [The author writes in a letter that L. Schwartz's thesis [Étude des sommes d'exponentielles réelles, Actual. Sci. Ind., no. 959, Hermann, Paris, 1943; these Rev. 7, 294] contains his results also.]

D. G. Bourgin (Urbana, Ill.).

Wintner, A. A class of Weierstrass bases. Quart. J. Math., Oxford Ser. 18, 209–214 (1947).

Consider $\phi(t) = \sum_1^\infty a_n \cos nt$, $\{a_n\} \in l_1$. Then $\{\phi(nt)\}$ is complete on $C(0, \pi)$ if $\sum a_k x_{nk} = 0$ has only the trivial solution $x_i = 0$ for $\{x_i\} \in m$. For this it is necessary that $a_1 \neq 0$ and sufficient that $(\sum a_n n^{-s})^{-1} = \sum b_n n^{-s}$ with $\{b_n\} \in l_1$. The argument is from the classical Riesz theorem. Some examples are given. *D. G. Bourgin* (Urbana, Ill.).

Ditkin, V. A. On the compieteness of a system of functions. Doklady Akad. Nauk SSSR (N.S.) **56**, 899–901 (1947). (Russian)

The author uses the results of a previous note [same vol., 779–782 (1947); these Rev. 9, 353] to deduce that, if L_n^2 denotes the space of functions $f(x)$ with norm given by $\|f\|^2 = \int_{-\infty}^{\infty}(1+x^2)^{-n}|f(x)|^2 dx$, a necessary and sufficient condition for the completeness in L_n^2 of the set of all translations $f(x+\lambda)$ is that there is no interval in which $F(t) = e^{-t}(a_0 + a_1 t + \cdots + a_{n-1}t^{n-1})$, where

$$F(t) = \int_{-\infty}^{\infty}(1+ix)^{-n} f(x) e^{ixt}\,dx.$$

R. P. Boas, Jr. (Providence, R. I.).

Pitt, H. R. A note on the representation of functions by absolutely convergent Fourier integrals. Proc. Cambridge Philos. Soc. 44, 8–12 (1948).

This paper gives a comparison of the local properties of absolutely convergent Fourier series, Fourier transforms and Fourier-Stieltjes transforms. *R. H. Cameron.*

***Campbell, George A., and Foster, Ronald M. Fourier Integrals for Practical Applications.** D. Van Nostrand Company, Inc., New York, 1948. 177 pp. $3.50.

These tables were issued in 1931 as Bell Telephone System Technical Publications, Mathematical and Physical Monographs, no. **B-584** [the present edition contains no reference to this earlier publication]. Their utility is well known to anyone having occasion to deal with Fourier transforms. There is a short introduction which describes, in particular, the use of the tables in solving transient problems. The tables contain 763 pairs of transforms.

R. P. Boas, Jr. (Providence, R. I.).

Broggi, Ugo. Notes on determining functions. Revista Unión Mat. Argentina 12, 225–237 (1947). (Spanish)

This paper contains three disconnected notes. The first discusses the function

$$\sum_{n=0}^{\infty}(-1)^n\{1 - \binom{n}{1}2^{-r} + \binom{n}{2}3^{-r} - \cdots + (-1)^n(n+1)^{-r}\}s^{n+1}$$

and its relationship to Laguerre polynomials. The second contains a proof of a formula expressing Bessel functions in terms of Laguerre functions [cf. G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, p. 98; these Rev. 1, 14]. The third gives a condition under which a bound for a Laplace integral on its abscissa of convergence implies the same bound in the half-plane of convergence. *R. P. Boas, Jr.*

Shen, Yu-cheng. The identical vanishing of the Laplace integral. Duke Math. J. 14, 967–973 (1947).

Der Verf. gibt einen neuen Beweis des Satzes von Lerch-Wintner mittelst des Satzes von Blaschke, wonach eine im Einheitskreis holomorphe Funktion mit genügend kleinem Wachstum und ∞ vielen Nullstellen, die nicht zu rasch gegen die Peripherie des Einheitskreises konvergieren, identisch verschwindet. Beim Beweis genügt die Voraussetzung der gewöhnlichen und nicht der absoluten Konvergenz des Integrals $\int_0^\infty e^{-st}F(t)dt$. Der Verf. gibt auch Verallgemeinerungen seines Satzes auf das Integral $\int_{-\infty}^{\infty} e^{-st}F(t)dt$.

W. Saxer (Zürich).

Varma, R. S. A generalisation of Laplace's transform. Current Sci. 16, 17–18 (1947).

The Whittaker or $W_{k,m}$-transform of $f(x)$ is

$$p\int_0^\infty (2xp)^{-\frac{1}{2}}W_{k,m}(2xp)f(x)dx.$$

For this functional transformation the author gives without proof theorems which are analogous to certain well-known theorems of the Laplace transform theory (the Parseval relation, and the effect of differentiation with respect to $\log x$ and $\log p$) and some more general theorems which either do not exist at all in the Laplace transform theory or have different forms (the relation between $\int_0^\infty f(x)x^{-1}dx$ and the corresponding integral of its transform, the Whittaker transform of a Laplace transform, and an expansion which flows from the multiplication theorem of Whittaker functions). Particular cases of this transformation, other than the Laplace transformation, have kernels given by modified Bessel functions of the third kind, Laguerre polynomials or parabolic cylinder functions. *A. Erdélyi.*

Fraser, W. C. G. An inversion formula for an integral related to Dirichlet series. Amer. Math. Monthly **54**, 586–588 (1947).

L'auteur démontre une formule pour l'inversion de l'intégrale

$$f(s)=\int_1^\infty t^{-s}d\alpha(t) \tag{1}$$

analogue à la formule de Phragmén [Acta Math. **28**, 351–368 (1904)]. Soit c l'abscisse de convergence de (1) et supposons que $\alpha(t)=o(t^{c+\epsilon})$ si $c\geqq 0$ et $\alpha(t)=O(1)$ si $c<0$, et encore que la fonction $\alpha(t)$ est de variation bornée dans chaque intervalle fini $(1, R)$, $\alpha(1)=0$. Alors pour chaque point t pour lequel il existe des limites à droite $\alpha(t+)$ et à gauche $\alpha(t-)$ on a

$$\lim_{n\to\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}t^n f(ns)=e^{-1}\alpha(t-)+(1-e^{-1})\alpha(t+).$$

N. Obrechkoff (Sofia).

Korenblyum, B. I. On the representation of functions of class L^p by singular integrals at Lebesgue points. Doklady Akad. Nauk SSSR (N.S.) **58**, 973–976 (1947). (Russian)

The problem of the convergence of singular integrals, that is, of integrals $\int_a^b \varphi_n(x, t)f(t)dt$, to the generating function $f(x)$ at the Lebesgue points of the latter, has been solved in two cases only, namely for f bounded [Lebesgue, Ann. Fac. Sci. Univ. Toulouse (3) **1**, 25–117 (1909)] and for $f\in L$ [Faddeeff, Rec. Math. [Mat. Sbornik] N.S. **1(43)**, 351–368 (1936)]. In the present paper the author gives necessary and sufficient conditions for the case $f\in L^p$, $1<p<\infty$. Given any $\varphi(t)\in L^q(c, d)$, $q>1$, we consider the number $\mu\geqq 0$ defined by the conditions

$$\int_{E(\mu)}|\varphi|^q dt\geqq\mu(d-c)\geqq\int_{E_1(\mu)}|\varphi|^q dt,$$

where $E(\mu)$ and $E_1(\mu)$ are, respectively, the sets of points of (c, d) at which $|\varphi|\geqq\mu$ or $|\varphi|>\mu$. If $(c, d)=\tau$, we shall write $\mu=\mu[\varphi(t), \tau, q]$. If $a<x<b$ and if x is a Lebesgue point of $f\in L^p$, $1<p<\infty$, then necessary and sufficient conditions are (1) $\lim_{n\to\infty}\int_\alpha^\beta\varphi_n(x, t)dt=1$ for every $\alpha<x<\beta$, $a\leqq\alpha<\beta\leqq b$; (2) $\sum_{i=1}^{\infty}(\mu_{ni}^{+}$ meas $\tau_i^{+}+\mu_{ni}^{-}$ meas $\tau_i^{-})$ is bounded in n, where

$$\tau_i^{+}=(x+2^{-i}(b-x), x+2^{-(i-1)}(b-x)),$$
$$\tau_i^{-}=(x-2^{-(i-1)}(x-a), x-2^{-i}(x-a)),$$
$$\mu_{ni}^{+}=\mu[\varphi_n(x, t), \tau_i^{+}, q],\ \mu_{ni}^{-}=\mu[\varphi_n(x, t), \tau_i^{-}, q],\ p^{-1}+q^{-1}=1.$$

A. Zygmund (Chicago, Ill.).

Levitan, B. On generalizations of positive definite and of almost periodic sequences. Doklady Akad. Nauk SSSR (N.S.) **58**, 977–980 (1947). (Russian)

The author refers frequently to previous papers [Rec. Math. [Mat. Sbornik] N.S. **16(58)**, 259–280 (1945); **17(59)**, 9–44, 163–192 (1945); these Rev. **7**, 254; **8**, 157]. Detailed proofs are omitted. He defines a "basis of continuous functions" as a sequence of linearly independent continuous functions $\{\omega_p(x)\}$, such that $\omega_p(x)\omega_q(x)=\sum_{r=0}^{\infty}c_{pqr}\omega_r(x)$, $\overline{\omega_p(x)\omega_q(x)}=\sum_{r=0}^{\infty}c_{pqr}^{*}\omega_r(x)$, where c_{pqr}, c_{pqr}^{*} are constants and the bar denotes the complex conjugate. Examples of such sequences are x^p, $e^{2\pi ipx}$, $\cos p\cos^{-1}x$. If $\{\varphi_i(x)\}$ is another such basis and $\omega_p(x)=\sum_{i=0}^{\infty}\alpha_{pi}\varphi_i(x)$, $\sum|\alpha_{pi}|<\alpha$, $\varphi_i(x)=\sum\beta_{iq}\omega_q(x)$, $\sum|\beta_i|<\beta$, $\varphi_i(x)\varphi_j(x)=\sum a_{ijk}\varphi_k(x)$, $\varphi_i(x)\varphi_j(x)=\sum a_{ijk}^{*}\varphi_k(x)$, $|a_{ijk}|<A$, $|a_{ijk}^{*}|<A^{*}$, then $c_{pqr}=\sum_{i,j,k}\alpha_{pi}\alpha_{qj}\beta_{kr}a_{ijk}$, $c_{pqr}^{*}=\sum_{i,j,k}\alpha_{pi}\bar{\alpha}_{qj}\beta_{kr}a_{ijk}^{*}$. The author calls $\{x_r\}$ a positive definite sequence with respect to the cubical matrix $|||c_{pqr}^{*}|||$ if, for all complex ξ_r,

$$\sum_{p=0}^{N}\sum_{q=0}^{N}\xi_p\bar{\xi}_q\sum_{r=0}^{\infty}c_{pqr}^{*}x_r\geqq 0.$$

Lemma. If $\{x_r\}$ is positive definite with respect to $|||c_{pqr}^{*}|||$, then $y_k=\sum\beta_{kr}x_r$ is positive definite with respect to $|||a_{ijk}^{*}|||$. If $y_k=\int_{-\infty}^{\infty}\varphi_k(x)d\sigma(x)$ with a bounded nondecreasing $\sigma(x)$, then $x_r=\int_{-\infty}^{\infty}\omega_r(x)d\sigma(x)$. This result yields applications to the above examples. Another example is the complete system $\{\omega_p(x)\}$ of characteristic solutions of $u''-q(x)u+\lambda^2u=0$, $u'(0)=u'(\pi)=0$. The system $\{\varphi_p\}=\{\cos px\}$ bears the above relation to $\{\omega_p(x)\}$.

The author calls $\{x_r\}$ almost periodic with respect to $|||c_{pqr}|||$ if $\sum_r c_{pqr}x_r$ is conditionally compact with respect to uniform convergence over the integers q. Lemma. If $\sum_p|\beta_{pi}|<b$, $\sum_k|a_{ijk}|<\infty$, then the almost periodicity of $\{x_r\}$ with respect to $|||c_{pqr}|||$ implies the almost periodicity of $y_k=\sum_r\beta_{kr}x_r$ with respect to $|||a_{ijk}|||$. The following theorem shows the close analogy of almost periodicity in the classical and in the present generalized sense. If $\{\omega_i(x)\}$ are characteristic functions of the Sturm-Liouville problem and c_{pqr} are as above, then there exist $\{\alpha_k\}$, $\{a_k\}$ and $|\rho_k^{(m)}|$ such that $0\leqq\alpha_k\leqq\pi$, $0\leqq\rho_k^{(m)}\leqq 1$, $\lim_m\rho_k^{(m)}=1$, $a_k=\lim_N N^{-1}\sum_0^N x_r\omega_r(\alpha_k)$, $\sigma_r^{(m)}=\sum_k\rho_k^{(m)}a_k\omega_r(\alpha_k)$, $\lim_m\sigma_r^{(m)}=x_r$, uniformly for all r.

František Wolf (Berkeley, Calif.).

Levitan, B. A generalization of positive definiteness and a generalization of almost periodic functions. Doklady Akad. Nauk SSSR (N.S.) **58**, 1593–1596 (1947). (Russian)

This paper follows up the continuous analogy of the previous paper, and Stieltjes integrals replace discrete sums. Attention is paid especially to the example borrowed from the Sturm-Liouville theory. The solution of $\varphi''-\rho(x)\varphi+\lambda\varphi=0$ such that $\varphi(0, \lambda)=1$, $\varphi'(0, \lambda)=0$ is considered as a generalization of $e^{i\lambda x}$, positive definiteness and almost periodicity of functions are generalized in close analogy to the previous paper and analoguous results are enunciated. Proofs are absent. *František Wolf* (Berkeley, Calif.).

Polynomials, Polynomial Approximations

de Bruijn, N. G. Inequalities concerning polynomials in the complex domain. Nederl. Akad. Wetensch., Proc. **50**, 1265–1272 = Indagationes Math. **9**, 591–598 (1947).

By systematic application of a simple argument based on the Gauss-Lucas theorem, the author gets a number of

results (special cases of some of which had been known before). We quote a few of them. (1) Let $P(z)$ and $Q(z)$ be two polynomials, the degree of P not exceeding that of Q; if $|P(z)| \leqq |Q(z)|$ on the boundary B of a convex region R, and if all the roots of Q are in $R+B$, then $|P'(z)| \leqq |Q'(z)|$ on B. (2) Let C be a circular region (i.e., a closed or open domain having a circle as its boundary) in the z-plane, and let S be any set in the w-plane. If $P(z)$ is a polynomial of degree n and if $P(z)=w \epsilon S$ for every $z \epsilon C$, then $n^{-1}\xi P'(z)+P(z)-n^{-1}zP'(z) \epsilon S$ for any $z \epsilon C$ and any $\xi \epsilon C$. It immediately leads to the Erdös-Lax theorem [Bull. Amer. Math. Soc. 50, 509–513 (1944); these Rev. 6, 61]: (3) if $P(z)$ is of degree n, has no roots for $|z| \leqq 1$, and if $|P(z)| \leqq 1$ for $|z| \leqq 1$, then $|P'(z)| \leqq \frac{1}{2}n$ there. This is the limiting case $p=\infty$ of the following general result. (4) If $P(z)$ is of degree n, has no roots for $|z| \leqq 1$, then

$$\int_0^{2\pi} |n^{-1}P'(e^{i\theta})|^p d\theta \leqq C_p \int_0^{2\pi} |P(e^{i\theta})|^p d\theta, \qquad p \geqq 1,$$

with $C_p = 2^{-p}\pi^{-1}\Gamma(\frac{1}{2}p+1)\Gamma(\frac{1}{2}p+\frac{1}{2})$. [For $p=2$, see Lax, loc. cit.] *A. Zygmund* (Chicago, Ill.).

Lipka, Stephan. Über einige Sätze von Zeichenwechsel. Math. Naturwiss. Anz. Ungar. Akad. Wiss. 60, 70–82 (1941). (Hungarian. German summary)

The author proves the following theorems. (1) Let $f(x)=a_0+a_1x+\cdots+a_nx^n$; the a's are real. Then if all complex roots of $f(x)$ satisfy $\pi-\pi/n \leqq \arg x \leqq \pi+\pi/n$ the number of positive roots of $f(x)$ is equal to the number of changes of signs of the a's. (2) If $f(x)$ has precisely k positive roots and the complex roots satisfy $\pi-\pi/(k+2) \leqq \arg x \leqq \pi+\pi/(k+2)$ then the a's have precisely k changes of sign. (3) Let $\xi \pm n$ be the only pair of complex roots, $p_1, \cdots, p_k$ the positive roots. Assume $\xi<0$ and $|\xi| \geqq 2(p_1+\cdots+p_k)$. Then the a's have precisely k changes of sign. (4) Assume that all roots lie in the angular domains $\pi-\pi/n<\arg x<\pi+\pi/n$, $-\pi/n<\arg x<\pi/n$. Then the number of roots in $-\pi/n<\arg x<\pi/n$ equals the number of changes of signs of the a's. Theorem 4 easily follows from the theorem of Obrechkoff [C. R. Acad. Sci. Paris 177, 102–104 (1923)]. In the proof, a theorem of Schoenberg [Math. Z. 32, 321–328 (1930)] plays a fundamental role. *P. Erdös.*

Neĭmark, Yu. I. On the problem of the distribution of the roots of polynomials. Doklady Akad. Nauk SSSR (N.S.) 58, 357–360 (1947). (Russian)

The author develops anew the Hurwitz type of criterion for the number of zeros of a complex polynomial $P_n(z)$ in the left and right half-planes. He sets $y^nP_n(ix/y)=f_n(x,y)+ig_n(x,y)$, $f_n(x,y)=\sum_{j=0}^n a_jx^{n-j}y^j$ and $g_n(x,y)=\sum_{j=0}^n b_jx^{n-j}y^j$, where the a_j and b_j are real. Denoting by $D(f_n, g_n)$ the greatest common divisor of f_n and g_n and defining $F_n=t_{11}f_n+t_{12}g_n$ and $G_n=t_{21}f_n+t_{22}g_n$ with the t_{jk} as real parameters for which $T=t_{11}t_{22}-t_{12}t_{21} \neq 0$, he notes that $D(F_n, G_n)=D(f_n, g_n)$. Hence the polynomial $Q_n(z)$ associated with F_n, G_n has the same number of zeros on the imaginary axis as $P_n(z)$ and the same number in the left and right half-planes as $P_n(z)$ if $T>0$ or as $P(-z)$ if $T<0$. By varying the t_{jk} continuously, he transforms $P_n(z)$ into a $P_{n-1}(z)$ with one less zero than $P_n(z)$ in the left half-plane if $a_0b_1<0$ or the right half-plane if $a_0b_1>0$. This operation, which may be repeated, successively reducing the degree of the polynomial, is equivalent to reducing the matrix of order $2n$,

$$A=\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \cdots a_n & 0 & 0 \cdots \\ b_0 & b_1 & b_2 & b_3 \cdots b_n & 0 & 0 \cdots \\ 0 & a_0 & a_1 & a_2 \cdots a_{n-1} & a_n & 0 \cdots \\ 0 & b_0 & b_1 & b_2 \cdots b_{n-1} & b_n & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot \cdot \cdot \cdot & \cdot & \cdot \cdot \cdot \end{bmatrix},$$

to one with only 0 below the main diagonal. Denoting by B_j the determinant formed from the first j rows and columns of the matrix A, the author finally shows that $P_n(z)$ has k zeros in the left half-plane and $n-k$ in the right half-plane provided k variations of sign occur in the sequence $1, B_2, -B_4, \cdots, (-1)^{n-1}B_{2n}$. [Reviewer's note. An important reference omitted is to H. Bilharz, Z. Angew. Math. Mech. 24, 77–82 (1944); these Rev. 7, 62.]

M. Marden (Milwaukee, Wis.).

Remez, E. Ya. On the character of convergence of the Pólya-Jackson process in the general case of continuous polynomials. Doklady Akad. Nauk SSSR (N.S.) 58, 1283–1286 (1947). (Russian)

Let $v_0(x)$, $v_1(x)$, $\cdots$, $v_n(x)$ be real-valued and continuous in $[a, b]$, and let $\Phi(x)=v_0+c_1v_1+\cdots+c_nv_n$, where $c_1, \cdots, c_n$ are arbitrary real numbers, so that the quantities $\delta_0[\Phi]=\max_{a \leqq x \leqq b}|\Phi(x)|$ and $\delta_m[\Phi]=[(b-a)^{-1}\int_a^b|\Phi|^m dx]^{1/m}$ depend on $c_1, \cdots, c_n$. Let Φ_0 and Φ_m denote functions realizing respectively the minimum of $\delta_0[\Phi]$ and of $\delta_m[\Phi]$ for all possible c's. If we set $\delta_0[\Phi_0]=\rho$, then $\delta_0[\Phi_m]-\rho=\alpha_m\rho$, where $\alpha_m \to 0$ as $m \to \infty$. The author shows that α_m may tend to 0 arbitrarily slowly. *A. Zygmund* (Chicago, Ill.).

Remez, E. Ya. Estimates of the rapidity of convergence of the Pólya-Jackson process for continuous polynomials with supplementary structural conditions. Doklady Akad. Nauk SSSR (N.S.) 58, 1601–1604 (1947). (Russian)

The author investigates the rapidity with which the numbers α_m [see the preceding review] tend to zero as $m \to \infty$, under the assumption that the moduli of continuity of all the functions $v_0(x)$, $v_1(x)$, $\cdots$, $v_n(x)$ are majorized by a fixed multiple of $\tilde{\omega}(\delta)$, which is increasing and concave. He shows that $\alpha_m=O[\varphi(m)]$, where the function φ is defined as follows. Let $\chi(x)$ be the function inverse to $\tilde{\omega}(x)$. If M is positive and sufficiently large, the equation $M=A^{-1}\log[1/\chi(A)]$ admits a unique solution $A=\varphi(M)$. The estimate $\alpha_m=O[\varphi(m)]$ is the best possible. Example: if $\tilde{\omega}(\delta)=\delta^\tau$ $(0<\tau \leqq 1)$, then $\varphi(m) \sim (\tau m)^{-1}\log m$.

A. Zygmund (Chicago, Ill.).

Remez, E. Ya. On the limiting process of Pólya-Jackson-Julia and certain corresponding interpolation algorithms. Doklady Akad. Nauk SSSR (N.S.) 58, 1901–1904 (1947). (Russian)

The problems here are similar to those discussed in the two notes reviewed above. Given a set E_N of N points $x_{N1}<x_{N2}<\cdots<x_{NN}$ in $[a, b]$, the author considers the maximum of Φ over E_N and the discrete average $[N^{-1}\sum_{j=1}^N|\Phi(x_{Nj})|^m]^{1/m}$, and then the minima (with respect to the c's) of these maxima. Under certain conditions of uniformity of the distribution of the points of E_N as $N \to \infty$ he obtains conclusions analogous to the preceding ones.

A. Zygmund (Chicago, Ill.).

Berman, D. L. On the interpolation process of Hermite. Doklady Akad. Nauk SSSR (N.S.) 58, 1569–1571 (1947). (Russian)

Given a triangular matrix of numbers $\{x_k^{(n)}\}$ $(n=1, 2, \cdots; 1 \leqq k \leqq n)$, all distinct for the same value of n and situated in

an interval $[a, b]$, let $\omega_n(x)=(x-x_1^{(n)})\cdots(x-x_n^{(n)})$,

$$v_k^{(n)}(x)=1-\frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}(x-x_k^{(n)}),$$

$$n=1, 2, \cdots; 1\leqq k\leqq n.$$

Fejér [Math. Ann. 106, 1–55 (1932)] calls the matrix $\{x_k^{(n)}\}$ normal if $v_k^{(n)}(x)>0$ for all n and k. The author shows that if (1) $\{x_k^{(n)}\}$ is normal, (2) $f(x)$ is a function continuous in $[a, b]$, (3) $P_1(x), \cdots, P_n(x), \cdots$ is a sequence of polynomials converging uniformly to $f(x)$ in $[a, b]$, the degree of P_n being at most $2n-1$, then the interpolating polynomials $H_n(x)$ of degree at most $2n-1$ defined by the conditions $H_n(x_k^{(n)})=f(x_k^{(n)})$, $H_n'(x_k^{(n)})=P_n'(x_k^{(n)})$ $(k=1, \cdots, n)$ converge uniformly to $f(x)$ in $[a, b]$. A somewhat similar theorem was proved by Grünwald [Acta Math. 75, 219–245 (1943); these Rev. 7, 157], who showed that, if $v_k^{(n)}(x)>\rho>0$ in $[a, b]$ for all n and k, if $f(x)$ is continuous in $[-1, 1]$, and if the numbers $d_k^{(n)}$ satisfy the inequalities $|d_k^{(n)}|<n^{\epsilon-1}$ for all n and k $(\epsilon>0)$, then the polynomials $H_n(x)$ of degree at most $2n-1$ and defined by the conditions $H_n(x_k^{(n)})=f(x)$, $H_n'(x_k^{(n)})=d_k^{(n)}$, converge uniformly to $f(x)$ in $[-1, 1]$. The author shows by an example that in certain cases his result is stronger than Grünwald's. *A. Zygmund.*

Tortrat, Albert. Sur les fonctions d'Hermite dérivées de l'exponentielle quadratique $e^{-\varphi(x_i)}$ et quelques équations intégrales. C. R. Acad. Sci. Paris 226, 298–300 (1948).

Discussion of the bilinear generating function of Hermite polynomials in several variables, without any essentially new results [cf. L. Koschmieder, Math. Z. 43, 783–792 (1938); A. Erdélyi, Math. Z. 44, 201–211 (1938)]. *A. Erdélyi* (Edinburgh).

Tortrat, Albert. Sur les fonctions orthogonales d'Hermite dérivées d'une exponentielle quadratique. Valeur asymptotique. C. R. Acad. Sci. Paris 226, 543–545; errata, 758–759 (1948).

The author obtains an estimate of Hermite polynomials of several variables which he uses to prove the convergence of the bilinear generating function [cf. the preceding review]. He also makes some remarks on the integration of the system of partial differential equations satisfied by these polynomials. *A. Erdélyi* (Edinburgh).

Korovkin, P. P. The asymptotic representation of polynomials orthogonal over a region. Doklady Akad. Nauk SSSR (N.S.) 58, 1883–1885 (1947). (Russian)

Let C be a closed analytic curve and d, D the interior and exterior regions determined by C, and let $x=\psi(z)=cz+\sum_0^\infty c_n/z^n$ $(c>0)$ be the function mapping the region $|z|>1$ conformally on D, with $z=\delta(x)$ as the inverse function ($\psi(z)$ is regular and single-valued for $|z|>\rho$, where $\rho<1$). Let $f(x)$ be analytic, and without zeros, in the region $|\delta(x)|>r\geqq\rho$, where $r<1$; and let $\rho(x)$ be a nonnegative summable function in d, with $\rho(x)=|f(x)\delta'(x)|^2$ for $r<|\delta(x)|<1$. The author is concerned with the problem of obtaining an asymptotic expression for the set of polynomials $q_n(x)$, $n=0, 1, \cdots$, uniquely defined by the orthogonality condition (over the region d) $\int_d\rho(x)q_n(x)\overline{q_m(x)}d\sigma=0$ or 1 according as $n\neq m$ or $n=m$ and the condition that the leading coefficient of $q_n(x)$, denoted by $1/\lambda_n$, is positive. For every polynomial $B_n(x)$ of degree n with leading coefficient unity, $\lambda_n^2\leqq\int_d\rho(x)|B_n(x)|^2d\sigma$.

In particular, choose $B_n(x)=\Phi_n(x, 1/f)$, the generalized Faber polynomial determined by $1/f$, whose leading coefficient is $(f(\infty)c^n)^{-1}$, and for which the following asymptotic relation holds: $\Phi_n(x, 1/f)=\delta^n(x)/f(x)+\epsilon_n[\delta(x)]$ $(|\delta(x)|>r)$, with $\epsilon_n[\delta(x)]=o(r_1^n)$, $r_1>r$ [Korovkin, Rec. Math. [Mat. Sbornik] N.S. 9(51), 469–485 (1941); these Rev. 3, 114]. Then $\lambda_n^2/[f^2(\infty)c^{2n}]\leqq\int_d\rho(x)|\Phi_n(x, 1/f)|^2d\sigma$. Write d as the sum of the ring S_{r_0}: $r_0\leqq|\delta(x)|<1$ $(r_0>r)$ and the region d_{r_0} determined by the curve $|\delta(x)|=r_0$, and take $r_1>r_0$. The integrals $\int_{d_{r_0}}$ and $\int_{S_{r_0}}$ are separately estimated to yield the respective results $o(r_1^{2n})$, $(\pi/(n+1)+o(r_1^{2n}))$, so $\lambda_n^2\leqq f^2(\infty)c^{2n}(\pi/(n+1)+o(r_1^{2n}))$. Again, from $1=\int_d\rho(x)|q_n(x)|^2d\sigma\geqq\int_{S_{r_0}}$ it is shown that $1\geqq(\pi/(n+1))f^2(\infty)c^{2n}\lambda_n^{-2}(1-r_0^{2n+2})$; so

$$\lambda_n^2=(\pi/(n+1))f^2(\infty)c^{2n}(1+o(r_1^{2n})).$$

The function $\varphi_n(z)=f[\psi(z)]q_n[\psi(z)]/z^n=f(\infty)c^n/\lambda_n+\sum_{k=1}^\infty|a_k^{(n)}|/z^k$ is regular in $|z|>r$, and it is shown that $\sum_{k=1}^\infty|a_k^{(n)}|^2=o(r_1^{2n})$, so (by Schwarz's inequality) $\sum_{k=1}^\infty|a_k^{(n)}/z^k|=o(r_1^n)$, $|z|>1$. This leads to the desired asymptotic relations $q_n(x)=((n+1)/\pi)^{\frac{1}{2}}\Phi_n(x, 1/f)+o(r_1^n)=((n+1)/\pi)^{\frac{1}{2}}\delta^n(x)/f(x)+o(r_1^n)$, $r_1>r$, valid in the region $|\delta(x)|>r$. From this follows the result: all functions regular in the closed region $\bar{d}_r$ can be represented, and uniquely, in series of the polynomials $q_n(x)$. *I. M. Sheffer.*

Special Functions

***Sales Vallés, Francisco de A. Contribución al Estudio de una Ley de Probabilidad (Primera Ley de Errores de Laplace). [Contribution to the Study of a Probability Law (First Law of Errors of Laplace)].** Thesis, University of Barcelona, 1947. 76 pp.

By the first Laplace law is meant the density function $e^{-2\beta|x|}/\delta$. The author studies its moments, orthogonal expansions, convolutions with itself, etc. Some considerations are generalized to the density $e^{-\alpha|x|-h\delta^{-\epsilon}}$. He also studies the problem of estimating the parameters from empirical material. *W. Feller* (Ithaca, N. Y.).

Matumura, Sôzi, und Kroll, Wolfgang. Über Flächen und Kurven. XLVI. Über die Nullstellen von $R_n=\int_{-1}^x P_n(x)dx$. Mem. Fac. Sci. Taihoku Imp. Univ. Ser. I. 1, 93–94 (1944).

[In the original, P was misprinted for R.] The authors find some properties of the zeros of R_n, in particular an asymptotic expression for large n. They do not appear to notice that R_n can be expressed in terms of ultraspherical polynomials (or associated Legendre functions) about whose zeros information is available. *A. Erdélyi.*

Kourganoff, V. Sur les fonctions $K_n(x)=\int^\infty\frac{e^{-xt}}{t^n}dt$ et certaines intégrales qui s'y rattachent. I. Propriétés générales des $K_n(x)$ et calcul des intégrales

$$I_{pm}(a)=\int_0^\infty e^{-px}x^mK_n(ax)dx.$$

Ann. Astrophysique 10, 282–299 (1947).

The function $K_n(x)$ plays an important part in astrophysics in the theory of radiative equilibrium of the stellar atmosphere, and also in nuclear physics in the theory of neutron diffusion [cf. also a report by G. Placzek, National Research Council of Canada, Division of Atomic Energy, Document no. MT-1 (1946); these Rev. 9, 159]. In this

paper the author derives the well-known properties of this function, including its connection with Prym's $Q(x, n)$, Whittaker's $W_{k,m}(x)$, Macrobert's generalised hypergeometric function, and for $n=1$ with the exponential integral and logarithmic integral function. The integral $I_{pns}(a)$ is discussed [cf. also C. R. Acad. Sci. Paris 225, 430–431 (1947); these Rev. 9, 91]. The coefficients in the expansion of $I_{pns}(a)$ for various values of p, n, s are given, also graphs of K_1, xK_1, K_2, $2K_3$, $\frac{1}{2}-2K_4$, $\frac{1}{2}x+2K_5$, and 4D tables for the last four functions in the range $x=0.00(0.01)0.20(0.02)2.00$.

A. Erdélyi (Edinburgh).

Bouwkamp, C. J. Concerning a new transcendent, its tabulation and application in antenna theory. Quart. Appl. Math. 5, 394–402 (1948).

This paper gives theory and tables connected with the function

$$E_1(z)+\int_0^z t^{-1}E(t)dt+\int_0^1\int_0^1 \frac{1-e^{-izst}}{st}dsdt$$

in which

$$E(z)=\gamma+\ln z-\mathrm{Ci}\,(z)+i\,\mathrm{Si}\,(z)=\int_0^z \frac{1-e^{-u}}{t}dt,$$

where Ci (z) and Si (z) are the familiar sine and cosine integrals. Series for $E_1(z)$ are given in powers of z and of $1/z$ (asymptotic). A series for $E_1(z+\Delta)-E_1(z)$ in powers of Δ is also developed.

The remainder of the theory concerns an expression derived by E. Hallén [Nova Acta Reg. Soc. Sci. Upsaliensis (4) 11, no. 4 (1938)] for the self-impedance of the centre-fed perfectly conducting cylindrical antenna, involving coefficients

$$\alpha_1=\tfrac{1}{2}e^{ix}E(4x)-\cos x\cdot E(2x),$$
$$\beta_1=\tfrac{1}{2}ie^{ix}\{E(4x)-4E(2x)\}+\sin x\{\ln 4-E(2x)\},$$
$$\alpha_2=-\alpha_1(x)\{\ln 4+E(2x)\}-\tfrac{1}{2}\cos x\cdot E^2(2x)$$
$$+2i\sin x\cdot E_1(4x)+\cos x\cdot\{E_1(4x)-2E_1(2x)\}.$$

The first two results are quoted and the last developed in the paper.

The tables give real and imaginary parts of $E_1(z)$ to 6 decimals for $z=0(0.2)20$, and real and imaginary parts of α_1 and α_2 to 4 decimals for $x=0(0.1)5$. Reference is made to earlier tables by C. J. Bouwkamp [Physica 9, 609–631 (1942); these Rev. 5, 163] and by R. King and F. G. Blake [Proc. I. R. E. 30, 335–349 (1942)]. *J. C. P. Miller.*

Bouwkamp, C. J. A study of Bessel functions in connection with the problem of two mutually attracting circular discs. Nederl. Akad. Wetensch., Proc. 50, 1071–1083 = Indagationes Math. 9, 485–497 (1947).

Two nonoverlapping homogeneous circular discs lie in the same plane. For interaction forces derived from the potential $\log r$, it is known that the mutual energy is exactly as if the masses of the discs were concentrated at their respective centers. The author raises the question of whether there are other potentials $V(r)$ for which the mutual energy is of the form $m(a, b)V(c)$, where a, b are the radii of the discs, c is the distance of their centers, and $m(a, b)$ is a reduced mass, independent of c. He shows that for any positive t, the potential $V(r)=K_0(rt)$ has this property. For a more involved potential of the type

$$V(r)=\int_0^\infty f(t)K_0(rt)dt$$

the mutual energy is

$$U(a, b; c)=\frac{4}{ab}\int_0^\infty I_1(at)I_1(bt)K_0(ct)f(t)t^{-2}dt.$$

The potential $V(r)=r^{-n}$ (n positive), which is of this form and of particular interest from the point of view of physical applications, is discussed in detail. In this discussion various integrals with Bessel functions arise and are evaluated in terms of hypergeometric functions. The cases in which one of the radii is zero, or both are equal, can be treated in full generality; for unequal nonvanishing radii the evaluation is carried through for small integer n. The interaction of a half-plane with a circular disc is also discussed.

A. Erdélyi (Edinburgh).

Seifert, H. Die hypergeometrischen Differentialgleichungen der Gasdynamik. Math. Ann. 120, 75–126 (1947).

The nonlinear partial differential equation for the velocity potential of a stationary two-dimensional compressible flow can be linearized either by Legendre's transformation or by the Molenbroek-Chaplygin transformation. In both cases the equation has normal solutions which are products of a function $\psi(\tau)$ (of a real variable τ which varies between 0 and 1) and of a trigonometric function; $\psi(\tau)$ satisfies a hypergeometric equation which has two linearly independent solutions, $\psi_n(\tau)=\tau^{\frac{1}{2}n}F(a_n, b_n; n+1; \tau)$ and $\psi_{-n}(\tau)$ (except when n is an integer when it is necessary to modify the second solution). Here $a_n+b_n=n+\epsilon(\gamma-1)^{-1}$, $a_nb_n=-\frac{1}{2}n(n-\epsilon)(\gamma-1)^{-1}$, $\gamma>1$, $\epsilon=\pm 1$. In order to investigate solutions of the partial differential equation which are infinite series of normal solutions, it is necessary to study the asymptotic behaviour of the solutions $\psi_{\pm n}(\tau)$ for large n. Investigations to this effect were begun by G. Guderley and are continued by the author [see also the following review].

The differential equation for ψ is of the form

$$\psi''(\tau)+\varphi(\tau)\psi'(\tau)-n^2q(\tau)\psi(\tau)=0;$$

$q(\tau)$ is positive in the subsonic region

$$0<\tau<\tau^*=(\gamma-1)(\gamma+1)^{-1}$$

and negative in the supersonic region $\tau^*<\tau<1$, and the asymptotic formulae are different in these two regions. The asymptotic formulae for ψ_n (n positive) in both the subsonic and supersonic regions can be obtained directly from the differential equation. For ψ_{-n}, however, it is necessary to represent the solution by a contour integral, and evaluate this integral by the method of steepest descent. Actually, the method of steepest descent is used by the author to find asymptotic representations of both ψ_n and ψ_{-n} in the subsonic region $0<\tau<\tau^*$, the supersonic region $\tau^*<\tau<1$, and also in the transsonic region where τ and τ^* are nearly equal. There is also a discussion of the zeros of $\psi_{\pm n}(\tau)$.

In the last section the author discusses the validity of the asymptotic formulae for complex τ. In this section he treats, more generally, $F(a, b; c; \tau)$, where a, b, c are real and of the order of n, and n is a large real parameter. *A. Erdélyi.*

Lighthill, M. J. The hodograph transformation in transsonic flow. II. Auxiliary theorems on the hypergeometric functions $\psi_n(\tau)$. Proc. Roy. Soc. London. Ser. A. 191, 341–351 (1947).

[For part I cf. the same vol., 323–341 (1947); these Rev. 9, 391.] Fundamentally, this paper deals with the same problem as the one reviewed above; but both the method and the range of validity of the results are different, although

many points of contact arise. Here τ is restricted to the range $0<\tau<1$, but n is regarded as a complex parameter, and the behaviour of $\psi_n(\tau)$ is investigated in the complex n-plane where (for $0<\tau\leqq 1$) $\psi_n(\tau)$ is a meromorphic function of n with the partial fraction expansion

$$\psi_n(\tau)\overset{=}{\sim}e^{ns}\left\{1+n\sum_{m=2}^{\infty}\frac{C_m e^{ms}\psi_m(\tau)}{n+m}\right\}$$

which may be regarded as an interpolation formula. This partial fraction expansion is proved for the subsonic region; the C_m are constants and s is an elementary function of τ. The asymptotic representations are obtained directly from the differential equation and from series expansions, without using integral representations. The zeros of $\psi_n(\tau)$ are also discussed. Complex values of τ are not envisaged.

A. Erdélyi (Edinburgh).

Ferguson, D. F., and Lighthill, M. J. The hodograph transformation in transsonic flow. IV. Tables. Proc. Roy. Soc. London. Ser. A. **192**, 135–142 (1947).

[Cf. the preceding review. Part III appeared in the same vol., 352–369 (1947); these Rev. **9**, 391.] To aid in the solution of compressible flow problems where extensions of certain incompressible flow patterns are desired, a set of tables are presented for the functions which occur in solutions obtained by the hodograph transformation. In particular, the hypergeometric functions $F_n(\tau)=F(a_n, b_n; n+1; \tau)$, where $a_n+b_n=n-(\gamma-1)^{-1}$; $a_nb_n=-n(n+1)/2(\gamma-1)$, and $F_n'(\tau)$; $\psi_n(\tau)=\tau^{n/2}F_n(\tau)$ and $\psi_n'(\tau)$, are tabulated for $n=1(1)15$, $\tau=0(0.02)0.05$; $\frac{1}{8}$. Tables are also given for c_m, $m=1(1)15$, and for $s(\tau)$, $\tau=0(0.02)0.16$; $\frac{1}{8}$; for definitions of c_m and $s(\tau)$ see part III. *N. A. Hall* (Minneapolis, Minn.).

Koschmieder, Lothar. Integrale mit hypergeometrischen Integranden. Acta Math. **79**, 241–254 (1947).

This is a lecture given by the author in 1943. It centres around integrals of the convolution type involving hypergeometric functions, and the application of fractional differentiation to these functions. The first part is expository, mainly historical and bibliographical. In it the author traces the history of the functional equation

$$\Gamma(\nu)\Gamma(\gamma-\nu)F(\alpha,\beta;\gamma;x)=\Gamma(\gamma)\int_0^1 s^{\nu-1}(1-s)^{\gamma-\nu-1}F(\alpha,\beta;\nu;sx)ds$$

and of related formulae, correcting erroneous statements in the literature regarding the history of this relation [but he fails to refer to relevant work by H. Bateman, Trans. Cambridge Philos. Soc. **21**, 171–196 (1909), in particular, p. 184]. The second part brings various extensions and generalisations to hypergeometric functions of two variables, and is based on original research. *A. Erdélyi* (Edinburgh).

Shanker, Hari. Parabolic cylinder functions which are Hankel-transforms of each other. J. Indian Math. Soc. (N.S.) **10**, 62–64 (1946).

The author uses operational calculus methods to prove that the functions

$$\psi(x)=\exp(x^2/4)x^{\nu+n-1}\sum_{r=0}^{m}{}_mC_r(-x)^rD_{-n-2\nu-r}(x),$$

$$\phi(x)=\exp(x^2/4)x^{\nu+m-1}\sum_{r=0}^{n}{}_nC_r(-x)^rD_{-m-2\nu-r}(x)$$

are J_ν-transforms of each other, where m, n are positive integers and $\Re(\nu-\frac{1}{2})>0$. Some special cases are briefly discussed. *M. C. Gray* (New York, N. Y.).

Auluck, F. C. Energy levels of an artificially bounded linear oscillator. Proc. Nat. Inst. Sci. India **7**, 133–140 (1941).

The Schrödinger equation of the quantum-mechanical system described in the title can be reduced to Weber's differential equation of parabolic cylinder functions [Whittaker and Watson, A Course of Modern Analysis, § 16.5] and the wave function must be an even or odd solution of this differential equation and vanish for a given value z_0 of the variable. The author shows that $D_n(z)+D_n(-z)$ has m positive zeros when n is between $2m-2$ and $2m$, and that $D_n(z)-D_n(-z)$ has m positive zeros when n is between $2m-1$ and $2m+1$. This supplements A. Milne's result [Proc. Edinburgh Math. Soc. (1) **32**, 2–14 (1914)] on the zeros of $D_n(z)$. From this result, and the known asymptotic formulae, the author obtains information about those values of n for which $D_n(z_0)\pm D_n(-z_0)$ vanishes. The energy levels themselves are multiples of $n+\frac{1}{2}$. *A. Erdélyi.*

Lebedev, N. N. An integral equation for the periodic solutions of the equation $u''+(a_0+a_1\cos 2x+a_2\cos 4x)u=0$. Doklady Akad. Nauk SSSR (N.S.) **59**, 25–28 (1948). (Russian)

In a preceding article [C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 391–394 (1946); these Rev. **8**, 269] the author proved that the periodic solutions of the above equation satisfy an integral equation

$$u(x)=\lambda\int_{-\pi}^{\pi}K(x,s)u(s)ds,$$

where K is any symmetric periodic solution of a partial differential equation of the second order; furthermore, he discussed certain types of functions K with the prescribed properties. In the present paper, he investigates other kernels, of the form

$$K=\exp\{(a_2/2)^{\frac{1}{2}}(\cos 2x+\cos 2s)\}\psi(x)\psi(s)F(\sigma),\qquad \sigma=\varphi(x)\varphi(s),$$

where φ and ψ are simple trigonometric functions and F is a hypergeometric function. Eight new different kernels are thus obtained.

Standard results for Mathieu's equation follow, in particular, if $a_1=0$ or $a_2=0$. If $a_1/4(2a_2)^{\frac{1}{2}}=\pm(n+\frac{1}{2})$ $(n=0, 1, 2, \cdots)$, the kernel reduces essentially to a Hermite polynomial and the periodic solutions have the form $u(x)=e^{\pm 2a_2\cos 2x}\cdot v(x)$, where $v(x)$ is a trigonometric polynomial.

J. L. Massera (Montevideo).

Campbell, Robert. Sur les développements en séries de Bessel des fonctions de Mathieu associées de période $2s\pi$. C. R. Acad. Sci. Paris **226**, 300–302 (1948).

The author expands the solution of the associated Mathieu equation in Bessel functions, obtaining a three-term recurrence relation among the coefficients. By employing the integral equation for the solution, relations between this solution and a solution in series of Gegenbauer polynomials are obtained. *H. Feshbach* (Cambridge, Mass.).

Selmer, Ernst S. Some rapidly converging series for the elliptic $\wp$-function of Weierstrass. Norske Vid. Selsk. Forh., Trondhjem **19**, no. 28, 112–115 (1947).

In order to improve the convergence of the power series for the $\wp$-function, the author uses the device of subtracting

from the function the meromorphic parts of a number of poles near the origin. In view of the increased region of convergence thus obtained, it may be expected that the series converges faster than before. In the two numerical examples worked out by the author this is indeed the case.
Z. Nehari (Cambridge, Mass.).

Harmonic Functions, Potential Theory

Gustin, William. A bilinear integral identity for harmonic functions. Amer. J. Math. **70**, 212–220 (1948).

Dans l'espace euclidien à $\tau \geqq 2$ dimensions, soient φ_1 harmonique dans D_1 ouvert, φ_2 harmonique dans D_2 ouvert, q_i et q_2 deux points respectifs de D_1, D_2 et x un vecteur-unité variable. Alors si p_1, p_2 sont positifs et inférieurs aux distances de q_1, q_2 aux frontières de D_1, D_2, l'intégrale $\int \varphi_1(q_1+p_1x)\varphi_2(q_2+p_2x)d\sigma_x$ (où $d\sigma$ est l'élément d'aire de la sphère-unité) ne dépend de p_1 et p_2 que par le produit p_1p_2. On en donne deux démonstrations puis des applications. On retrouve par exemple la nullité d'une fonction harmonique dans un domaine si elle est nulle dans une partie ouverte, ce qui était jusqu'ici déduit de l'analyticité.
M. Brelot (Grenoble).

Monna, A. F. Sur une formule d'inversion de Stieltjes et la théorie du potentiel. Nederl. Akad. Wetensch., Proc. **50**, 1056–1062 = Indagationes Math. **9**, 470–476 (1947).

Let $R(z)=\int_{\gamma=0}(z-t)^{-1}dm(\epsilon_t)$, where $z=x+iy$ and $m(\epsilon)$ is a distribution of masses of variable sign on $y=0$. Here $R(z)$ is holomorphic for $y\neq 0$; its imaginary part is

$$\Im[R(z)]=-\int_{\gamma=0}\left[\frac{\partial}{\partial\epsilon}(\log(1/r_\epsilon))\right]_{\epsilon=0}dm(\epsilon_t),$$

where $r_\epsilon=\{(x-t)^2+(y-\epsilon)^2\}^{\frac{1}{2}}$. The integral on the right suggests the study of a potential $U(P)$ analogous to that due to a double distribution:

$$U(P)=\int_\Gamma \frac{\partial}{\partial n}\log r^{-1}dm(\epsilon),$$

where $m(\epsilon)$ is a distribution of masses on the boundary Γ of a plane region and n is the normal to Γ in a suitably chosen sense. The author finds that if Γ is the x-axis then

$$\lim_{y\to+0}\int_{x_1}^{x_2}U(z)dx=\pi\int_{x_1}^{x_2}dm(\epsilon_x)+\tfrac{1}{2}\pi(m_1+m_2),$$

where m_1 and m_2 are the jumps in $m(\epsilon_x)$ at x_1 and x_2, respectively. A combination of this with a similar formula in which $y\to-0$ yields

$$\lim_{y\to+0}\int_{x_1}^{x_2}U(z)dx-\lim_{y\to-0}\int_{x_1}^{x_2}U(z)dx=2\pi m(e_{x_1,x_2})+\pi(m_1+m_2),$$

where e_{x_1,x_2} represents the interval $x_1<x<x_2$. Corresponding formulas are obtained for the case in which Γ is a circle. An application of these formulas is given and possible generalizations are suggested.
F. W. Perkins.

Koschmieder, Lothar. Eigenschaften harmonischer Reihen mit zeichenfester Summe in Räumen höherer Stufenzahl. Comment. Math. Helv. **21**, 44–57 (1948).

Dans la sphère de centre O, de rayon 1, de l'espace à $N+2$ dimensions, on considère le développement d'une fonction harmonique positive sous la forme (1) $\sum_{n=0}^{\infty}H_n(m)r^n$ $(0\leqq r<1)$, m représentant un point à la surface de la sphère unité; l'auteur, généralisant un résultat de Fejér [Monatsh. Math. Phys. **35**, 305–344 (1928)], établit que les sommes partielles de (1) sont positives dans la sphère de centre O de rayon $(N+2)^{-1}$. Les coefficients satisfont à

$$|H_n(m)|\leqq(2\mu+N)\Gamma(\mu+N)/\{\Gamma(N+1)\Gamma(\mu+1)\};$$

les sommes partielles s'annulent au plus $N+1$ fois sur un rayon.
P. Lelong (Lille).

Walker, A. G. Note on pseudo-harmonic functions. II. J. London Math. Soc. **22**, 101–104 (1947).

The author proves the following theorem, which is the converse of a mean value theorem contained in part I [same J. **20**, 33–39 (1945); these Rev. **7**, 448]. Let v be a function which has continuous second derivatives in a domain D of a three-dimensional space of constant curvature. Let M_sv denote the mean value of v over the sphere S of geodesic radius s and center O, and v_0 the value of v in O. Then the following condition is sufficient for v to satisfy in D the equation $\nabla^2v=\lambda v$, where ∇^2 denotes the second Beltrami differential operator: there exists a function $w(s)\sim 1+\lambda s^2/6$ for small s such that $M_sv=v_0w(s)$.
E. H. Rothe.

Deny, Jacques, et Lelong, Pierre. Étude des fonctions sousharmoniques dans un cylindre ou dans un cône. Bull. Soc. Math. France **75**, 89–112 (1947).

Développement d'une précédente note [C. R. Acad. Sci. Paris **224**, 1046–1048 (1947)], dont une analyse détaillée a paru dans ces Rev. **8**, 514.
H. Cartan (Paris).

Bourion, Georges. Sur l'indicatrice de croissance d'une fonction sousharmonique dans un angle. Bull. Sci. Math. (2) **71**, 17–25 (1947).

The Pólya growth indicatrix [Math. Z. **29**, 549–640 (1929), in particular, pp. 581–597] is treated for subharmonic functions.
M. Heins (Providence, R. I.).

Differential Equations

Dehousse, L. Sur le théorème d'Auric et son extension à un système différentiel linéaire, homogène, à coefficients constants. Bull. Soc. Roy. Sci. Liége **16**, 102–107 (1947).

Simple expressions for power series solutions are obtained and verified by using recursion formulas.
P. Franklin.

Predonzan, Arno. Sulle vibrazioni forzate di un sistema non dissipativo a due gradi di libertà. Ann. Scuola Norm. Super. Pisa (2) **12** (1943), 173–183 (1947).

The author studies the system of equations

$$b_{11}\ddot{q}_1+b_{12}\ddot{q}_2+a_{11}q_1+a_{12}q_2=k_1\sin wt,$$
$$b_{12}\ddot{q}_1+b_{22}\ddot{q}_2+a_{12}q_1+a_{22}q_2=k_2\sin wt,$$

under various restrictions on the coefficients b_{ij}, a_{ij}; $i, j=1, 2$.
R. Bellman (Stanford University, Calif.).

Nikitin, V. P., Turkin, V. K., and Kunickiĭ, N. P. On the construction of a diagram of the frequency of oscillation. Doklady Akad. Nauk SSSR (N.S.) **59**, 57–60 (1948). (Russian)

The authors are concerned with the problem of constructing an algorithm showing the relation between the frequency of an oscillation and the coefficients in the differential equations representing the motion of a system which tends to a

stationary state. A diagram of the frequency of oscillation is defined as a diagram which illustrates the dependence of the characteristic numbers of the functions entering in the solutions of the differential equations upon the coefficients in the differential equations. The problem of finding the characteristic numbers is reduced to solving an algebraic equation of the form $s^n+c_1s^{n-1}+c_2s^{n-2}+\cdots+c_{n-1}s+c_n=0$, where $c_k=a_k+ib_k$ $(k=1, 2, \cdots, n)$. According to the results obtained by E. Frank [Bull. Amer. Math. Soc. 52, 890–898 (1946); these Rev. 8, 154], the real parts of all the roots of the equation are negative when certain determinants formed from a_k and b_k are all positive. The region in which these conditions are satisfied is also the region in which the frequency of oscillation is never less than a constant q which enters in the system of differential equations.

S. D. Zeldin (Cambridge, Mass.).

Wintner, Aurel. On the solutions of radial wave equations. Physical Rev. (2) 73, 91–92 (1948).

In an earlier paper [Amer. J. Math. 69, 87–98 (1947); these Rev. 8, 381] the author proved that, if $-f$ is completely monotone in $(0, \infty)$, then the equation (1) $y''+fy=0$ has a completely monotone solution. This result is applied to a discussion of the radial wave equation

$$(2) \qquad (r^2R')'+[W_0-l(l+1)r^{-2}-V(r)]r^2R=0$$

which, upon setting $y=rR$, takes on the form (1) with $f(r)=W_0-l(l+1)r^{-2}-V(r)$. Thus if $W_0\leqq 0$ and $V(r)$ is completely monotone in $(0, \infty)$, then (1) and hence also (2) have completely monotone solutions. The author points out that the physically interesting special cases $V(r)=r^{-\alpha}$ or $e^{-\alpha r}$, $\alpha\geqq 0$, are completely monotone functions and that the latter functions form a ring under addition and multiplication. There are a couple of misleading misprints: $W_0\geqq 0$ instead of $W_0\leqq 0$ and nonincreasing instead of nondecreasing.

E. Hille (New Haven, Conn.).

Cafiero, Federico. Sull'approssimazione mediante poligonali degli integrali del sistema differenziale: $y'=F(x, y)$, $y(x_0)=y_0$. Giorn. Mat. Battaglini (4) 1(77), 28–35 (1947).

Under the single assumption that $F(x, y)$ is continuous in a rectangle, this paper develops a method of polygonal approximation which yields every solution of the system specified in the title. The method generalizes that of Cauchy-Lipschitz and in this respect differs from that of E. Bajada [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 261–268 (1947); these Rev. 9, 92]. *J. M. Thomas.*

***Friedrichs, K. O. Criteria for the discrete character of the spectra of ordinary differential operators.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 145–160. Interscience Publishers, Inc., New York, 1948. $5.50.

Let r, p, q, be functions of the variable ξ which ranges over the interval $\xi_-<\xi<\xi_+$, where ξ_- and ξ_+ may be finite or infinite. Let D denote the operator of differentiation with respect to ξ, and $L=r^{-1}[DpD-q]$. Simple sufficient criteria are given that the spectrum of the operator L (subject to appropriate boundary conditions or conditions taking the place of boundary conditions) be discrete, discrete below a finite value, or nondiscrete. These criteria depend solely on the behavior of the expression $r^{-1}[q+(4ph^2)^{-1}]$, where h is a function which is easily obtained in terms of integrals over p^{-1}. The criteria are applied to the operator $L=-D^2+q$, to operators connected with the Hermite, Legendre, Laguerre and Bessel functions, and also to the potential and Schroedinger operators for functions of n variables specialized to functions depending only on the distance ξ from the origin. *E. H. Rothe* (Ann Arbor, Mich.).

Ditkin, V. A. On the theory of a differential operator. Doklady Akad. Nauk SSSR (N.S.) 56, 779–782 (1947). (Russian)

The author studies the Hilbert space L_n^2 of functions $f(x)$ with norm given by $\|f\|^2=\int_{-\infty}^{\infty}|f(x)|^2(1+x^2)^{-n}dx$. He obtains formulas for $H(D)f$, where $D=id/dx$ and H has an nth derivative, in terms of $F(t)=\int f(x)(1+ix)^{-n}e^{ixt}dt$, and discusses the differential equation $H(D)f=0$. For example, if $H(x)\equiv 0$ in $a\leqq x\leqq b$ and $H(x)\neq 0$ outside (a, b), the general solution is

$$f(x)=(1+ix)^n\int_a^b G(t)e^{-ixt}dt+e^{ax}\sum_{k=0}^{n-1}a_kx^k,$$

where $G(t)\varepsilon L^2(a, b)$. The following theorem is also announced. If $\limsup_{r\to\infty}\|D^rf\|^{1/r}\leqq a$, then there is a $G(t)\varepsilon\ L^2$ such that

$$f(x)=(1+ix)^n\int_{-a}^{a}G(t)e^{-ixt}dt+e^{iax}\sum_{k=0}^{n-1}a_kx^k.$$

R. P. Boas, Jr. (Providence, R. I.).

Kalafati, P. On the Green's functions of ordinary quasi-differential equations. Doklady Akad. Nauk SSSR (N.S.) 59, 427–430 (1948). (Russian)

The author proves the following theorem. Given a differential system consisting of the equation

$$L(y)=\rho_0\frac{d}{dx}\rho_1\frac{d}{dx}\cdots\rho_{n-1}\frac{d}{dx}\rho_n y=0,$$

where $\rho_k(x)>0$ $(a\leqq x\leqq b;\ k=0, 1, \cdots, n)$, and the boundary conditions

$$\sum_{k=1}^{n}\alpha_{ik}D^{k-1}y(a)+\sum_{k=n+1}^{2n}\alpha_{ik}D^{k-n+1}y(b)=0,$$

$i=1, \cdots, n$, where $D^0y=\rho_0y$ and

$$D^ky=\rho_{n-k}\frac{d}{dx}\rho_{n-k+1}D^{k-1}y,$$

$k=1, \cdots, n$, with the matrix

$$A=\|\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{in}, \alpha_{i\,n+1}, \cdots, \alpha_{i\,2n}\|_{i=1}^{n},$$

let the matrix S be regular with respect to the numbers $q_1, \cdots, q_n, p_1, \cdots, p_n$. Then, if the matrix

$$T_1=A*S(q_1, \cdots, q_n, p_1, \cdots, p_n)$$

(where the symbol $*$ indicates the operation by which T_1 is obtained from A by means of S) is an ordinary even or odd K-matrix, the Green's function of the given differential system is an ordinary even or odd Kellogg kernel.

S. D. Zeldin (Cambridge, Mass.).

Scorza Dragoni, Giuseppe. Rettifica alla memoria: A proposito di alcuni teoremi sulle equazioni differenziali. Rend. Sem. Mat. Univ. Padova 16, 1–2 (1947).

The paper appeared in the same Rend. 15, 60–131 (1946); these Rev. 8, 207.

Germay, R.-H. Extension d'un théorème de Lagrange aux systèmes complètement intégrables d'équations aux différentielles totales de forme linéaire et homogène. Bull. Soc. Roy. Sci. Liége 16, 17–23 (1947).

Let

$$(*)\quad dz_j=\sum_{s=1}^{n}\{A_\mu^{(1)}(x_1,\cdots,x_n)z_1+\cdots+A_\mu^{(p)}(x_1,\cdots,x_n)z_p\}dx_s,\qquad j=1,\cdots,p,$$

be a completely integrable pth order system of total differential equations. The author shows that if one knows $k<p$ distinct integrals $z_{11},\cdots,z_{1p};\cdots;z_{k1},\cdots,z_{kp}$ of (*), the system (*) can be reduced to a completely integrable $(p-k)$th order system plus certain quadratures.

F. G. Dressel (Durham, N. C.).

Papy, Georges. Sur la divisibilité des formes alternées par des formes quadratiques régulières dans un espace à $2n$ dimensions. Bull. Soc. Roy. Sci. Liége 16, 24–30 (1947).

Let Ω be a symbolic (alternée) form of degree n in $2n$ indeterminates $x_1,\cdots,x_n,y_1,\cdots,y_n$, and $\Gamma=x_1y_1+\cdots+x_ny_n$. Lepage [same Bull. 15, 21–31 (1946); these Rev. 8, 499] showed that the equation $\Omega=\Theta_{n-2}\Gamma$ has a solution in the forms of degree $n-2$ in the same indeterminates, if and only if $\Omega_n\cdot w_1\cdots w_n=0$ for all systems of n forms $w_i=y_i-p_{ij}x_j$, where p_{ij} are arbitrary numbers ($p_{ij}=p_{ji}$). The present paper proves that the above theorem holds if the degree of Ω is any of the numbers $2,3,\cdots,n$. *F. G. Dressel.*

Saltykow, N. Systèmes d'équations aux différentielles totales complètement et non complètement intégrables. Bull. Soc. Math. France 75, 27–30 (1947).

The author supplies the solutions for two examples both of which were incorrectly treated by their authors [cf. E. Goursat, Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre, 2d ed., Paris, 1921, p. 133]. The first is due to Valyi and the solution is given in finite form. The second is due to Bourlet and the solution is obtained in implicit form in terms of two elliptic integrals of the first kind. *M. S. Knebelman* (Pullman, Wash.).

Vessiot, Ernest. Sur l'intégration des systèmes de Pfaff linéaires et homogènes à coefficients constants. C. R. Acad. Sci. Paris 226, 289–291 (1948).

This is a further discussion of a problem considered by N. Saltykow [same C. R. 225, 520–521 (1947); these Rev. 9, 186] and the reviewer's criticism of that paper applies to the present paper even more strongly. The problem is to determine the conditions under which the equations $\partial y^i/\partial t^\lambda=\sum_j A_\lambda^i y^j$ will admit a solution of the form $y^i=c^i\cdot e^{\theta_\lambda t^\lambda}$, $i,j=1,\cdots,n$; $\lambda=1,\cdots,p$. The author claims that there is an implicit assumption of complete integrability, which is not the case. All that is needed for this type of solution (writing the equations in matrix form) is that $(\sum_\lambda A_\lambda-I\cdot\theta_\lambda)c=0$; this obviously yields $(\sum_{\lambda,\mu}A_\lambda A_\mu-\sum_{\mu,\lambda}A_\mu A_\lambda)c=0$ which does not imply the vanishing of the commutator $\sum_{\lambda,\mu}A_\lambda A_\mu-\sum_{\mu,\lambda}A_\mu A_\lambda$, which is necessary for complete integrability. *M. S. Knebelman.*

Vessiot, Ernest. Sur la réductibilité des équations aux dérivées partielles du 1er ordre, à une inconnue, qui ne la contiennent pas et sont linéaires et homogènes par rapport à ses dérivées. Bull. Soc. Math. France 75, 9–26 (1947).

The general theory previously developed by the author [Ann. Sci. École Norm. Sup. (3) 63, 1–22 (1946); these Rev. 8, 584] is here applied to a single equation of the type indicated in the title. For each domain of rationality $\mathfrak{D}$ and each fundamental set of solutions $\mathfrak{F}$ there is defined a group $\mathfrak{G}$. If a rational function R of $\mathfrak{F}$ with coefficients in $\mathfrak{D}$ has values in $\mathfrak{D}$, it admits the transformations of $\mathfrak{G}$; and the converse is true if $\mathfrak{F}$ is primitive in a specifically defined sense. *J. M. Thomas* (Durham, N. C.).

Drach, Jules. Sur des équations aux dérivées partielles du premier et du second ordre dont les caractéristiques sont lignes asymptotiques des surfaces intégrales. C. R. Acad. Sci. Paris 226, 287–288 (1948).

This is a continuation of a paper with similar title [same C. R. 225, 1221–1224 (1947); these Rev. 9, 286]. The differential equations here considered are $r+2s\alpha+t\alpha^2=0$, $\lambda^2(rt-s^2)+1=0$, where λ and α are functions of x, y, p, q, the equations possessing a single first integral $V(x, y, p, q)=\varphi=$constant. The problem is solved under certain restrictions and illustrated by two examples.

M. S. Knebelman (Pullman, Wash.).

Baiada, Emilio. Sul teorema d'esistenza per le equazioni alle derivate parziali del primo ordine. Ann. Scuola Norm. Super. Pisa (2) 12 (1943), 135–145 (1947).

Let $f(x, y, z, q)$, f_y, f_z, f_q be continuous and uniformly bounded functions for $0\leqq x\leqq a$, and y, z, q unrestricted. Let the function $w(y)$ be continuous and bounded together with its first derivative. Assume also that f_y, f_z, f_q satisfy Lipschitz conditions in y, z, q, and $w'(y)$ satisfies a Lipschitz condition in y. The author presents a new proof of the known theorem that the Cauchy problem $\partial z/\partial x=f(x, y, z, \partial z/\partial x)$, $z(0, y)=w(y)$, $0\leqq x\leqq a$, with z_x, z_y continuous, has a solution.

F. G. Dressel (Durham, N. C.).

Baiada, Emilio. Sur problema di Cauchy per le equazioni alle derivate parziali. Ann. Scuola Norm. Super. Pisa (2) 12 (1943), 185–188 (1947).

The paper shows that the Cauchy problem $z_x=f(x, y, z, z_y)$, $z(0, y)=0$, may fail to have a solution with $\partial z/\partial x$, $\partial z/\partial y$ continuous, if the only assumption on $f(x, y, z, q)$ is that it is continuous and the first partial derivatives of f exist.

F. G. Dressel (Durham, N. C.).

Myškis, A. On the method of A. Haar in a question of the uniqueness of the solution of the problem of Cauchy for a system of partial differential equations. Doklady Akad. Nauk SSSR (N.S.) 58, 21–24 (1947). (Russian)

The author extends a method of proving the uniqueness of the solution of the Cauchy problem for the equation

$$\frac{\partial u}{\partial x_1}=\sum_{i=2}^{n}a_i(x_1,\cdots,x_n)\frac{\partial u}{\partial x_i}+b(x_1,\cdots,x_n)u+f(x_1,\cdots,x_n),$$

to prove uniqueness of solution of the Cauchy problem for the system

$$\frac{\partial u_s}{\partial x_1}=\sum_{j=1}^{k}\sum_{i=2}^{n}a_{ijs}(x_1,\cdots,x_n)\frac{\partial u_j}{\partial x_1}+\sum_{j=1}^{k}b_{js}(x_1,\cdots,x_n)u_j+f_s(x_1,\cdots,x_n),\quad s=1,\cdots,k.$$

R. Bellman (Stanford University, Calif.).

***Rellich, Franz. Das Eigenwertproblem von $\Delta u+\lambda u=0$ in Halbröhren.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 329–344. Interscience Publishers, Inc., New York, 1948. $5.50.

The main part of the paper is concerned with the spectrum of the eigenvalue problem

$$(1) \qquad \sum_{i=1}^{h} \partial^2 u/\partial x_i^2+\lambda u=0,$$

$u=0$ on the boundary, for a "half tube." The simplest example of a half tube is the cylinder $\sum_{i=1}^{h-1} x_i^2<a^2$, $x_h>0$; for the exact definition we refer to the paper. Previously [Jber. Deutsch. Math. Verein. 53, 57–65 (1943); these Rev. 8, 204] the author had shown that (1) has a purely continuous spectrum if the exterior normal at no point of the boundary F of the half tube G forms an angle less than 90° with the positive x_h-axis. In the present paper it is shown that the spectrum of (1) is discrete if the half tube G becomes "infinitely narrow," where the term "infinitely narrow" is defined in the following way: let E_ρ be the intersection of G with the plane $x_h=\rho$, and e_ρ the smallest eigenvalue of the $(h-1)$-dimensional problem $\sum_{i=1}^{h-1} \partial^2 u/\partial x_i^2+\lambda u=0$, $u=0$ on the boundary of E_ρ. Then G is called infinitely narrow if $\lim_{\rho\to\infty} e_\rho=+\infty$. (Except in the case $h=2$, this does not imply that the volume of E_ρ approaches 0 as ρ approaches ∞.) The proof is based on the well-known "Auswahlsatz" of the author [see, e.g., Courant and Hilbert, Methoden der mathematischen Physik, v. 2, Springer, Berlin, 1937, pp. 489, 519]. A case intermediate between the previously treated case and the case of an infinitely narrow tube is also treated. *E. H. Rothe.*

Birindelli, Carlo. Nuova trattazione di problemi al contorno di una striscia per l'equazione di Laplace in due variabili. Ann. Mat. Pura Appl. (4) 25, 155–195 (1946).

The problem is to find a function $u(x, y)$ satisfying $\Delta u=f(x, y)$ in the infinite strip T, bounded by $y=0$ and $y=a>0$, such that for $y=0$, $\alpha_0 u+\beta_0 u_y=\gamma_0(x)$ and for $y=a$, $\alpha u+\beta u_y=\gamma(x)$, $\gamma_0(x)$ and $\gamma(x)$ being given continuous functions; $f(x, y)$ is a given continuous function of x and y in T and on the boundary FT, satisfying moreover a Hölder condition; u and u_y are to be continuous in $T+FT$ and u_x, u_{xx}, u_{yy} in T; α_0, β_0, α, β are given constants such that $\alpha_0^2+\beta_0^2=\alpha^2+\beta^2=1$; in the first part it is supposed that all four are different from zero, and that $\alpha\beta_0<0$, $\alpha\beta>0$. Nothing being postulated on the behavior of u_x, u_{xx}, u_{yy} on FT, the investigation begins in a strip bounded by $y=\epsilon$ and $y=a-\epsilon$ $(0<2\epsilon<a)$. Afterwards the limit for $\epsilon\to 0$ is considered. The principal aim is to formulate conditions for the behavior of $u(x, y)$ if $x\to\infty$, sufficient to insure the uniqueness of the solution. Such a condition is $\lim_{|x|\to\infty} u(x, y)e^{\lambda_0^{\frac{1}{2}}|x|}=0$, uniformly with respect to y in the interval $(0, a)$. Here λ_0 is the least eigenvalue of the problem $v_{yy}+\lambda v=0$, $(\alpha_0 v+\beta_0 v_y)_{y=0}=0$, $(\alpha v+\beta v_y)_{y=a}=0$. In the theorems stated, sufficient conditions are also formulated for $f(u, y)$, $\gamma(u)$ and $\gamma_0(u)$. The solution is obtained in the form of a series $\sum v_n(y)u_n(x)$, where the index n refers to the eigenvalue λ_n of the problem, mentioned above. It is proved that the series is absolutely convergent in $T+FT$, uniformly convergent in each finite domain in $T+FT$, and that the function so determined satisfies all the requirements of the problem. In the second part the cases are treated where the constants α_0, β_0, α, β are not all different from zero. Here are included the problems of Dirichlet and Neumann and the mixed Dirichlet-Neumann problem. In this part only the results are recorded; for the proofs reference is made to the analogous cases in the first part. *H. Bremekamp.*

Ribaud, Gustave. Une solution nouvelle de l'équation de Fourier. C. R. Acad. Sci. Paris 226, 140–142 (1948).

Ribaud, Gustave. Le problème du mur indéfini avec flux calorifique constant. C. R. Acad. Sci. Paris 226, 204–206 (1948).

Ribaud, Gustave. Développements sur une solution de l'équation de Fourier dans le cas du mur d'épaisseur infinie. C. R. Acad. Sci. Paris 226, 449–451 (1948).

A family of functions of the type $t^m f(x/t^k)$ which satisfy the simple heat equation is pointed out here. The author shows how these functions can be used to obtain the formulas for the temperatures in a semi-infinite solid $x\geqq 0$, initially at temperature zero, when the flux through the boundary $x=0$ is a prescribed polynomial function of the time. The results are easily obtainable in the same form by direct procedures such as the method of Laplace transforms, but the indirect process used here involves only elementary calculus. *R. V. Churchill* (Ann Arbor, Mich.).

Rosovskiĭ, M. I. On certain thermal processes in solid bodies. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 17, 998–1004 (1947). (Russian)

This is a study of heat distribution in a solid body in the presence of hereditary effects. The author states that the method used is a direct application of Volterra's principle [Acta Math. 35, 295–356 (1912)]. The nonstationary temperature distributions in a homogeneous isotropic parallelepiped and sphere are determined under simple boundary conditions. The case of a stationary temperature in a solid body bounded by an arbitrary closed surface is also considered. It is shown that under certain restrictions this problem can be reduced to the solution of an integro-differential equation for a stationary electromagnetic field with hereditary phenomena. It is pointed out that the latter problem was solved by Volterra in the above mentioned paper. *H. P. Thielman* (Ames, Iowa).

Kriszten, Adolf. Funktionentheorie und Randwertproblem der Diracschen Differentialgleichungen. Comment. Math. Helv. 20, 333–365 (1947).

Im Anschluss an eine Arbeit von Fueter [Comment. Math. Helv. 16, 19–28 (1944); diese Rev. 5, 261] entwickelt der Verfasser eine neue Methode zur Integration der Diracschen Differentialgleichungen des Elektrons für den kräftefreien Fall, die wohl der Ausdehnung auf allgemeinere hyperbolische Systeme fähig ist. Es wird das Problem zugrundegelegt, wo die Randwerte der Diracschen Funktionen auf einer räumlich orientierten Hyperfläche Σ gegeben sind. Während Fueter eine lineare Integralgleichung für die Lösung ableitet, gelingt es dem Verfasser, das Problem explicite zu lösen. Seine Grundidee ist die Anwendung der Hadamardschen Methode der "partie finie" [Le Problème de Cauchy et les Équations aux Dérivées Partielles Linéaires Hyperboliques, Hermann, Paris, 1932].

In zwei Cliffordschen Algebren, deren Einheiten mit c_i und e_i bezeichnet sind, werden c- bzw. e-reguläre Funktionen definiert. Die Regularitätsbedingungen für die e-Funktionen sind mit den Diracschen Gleichungen identisch, die jenigen für die c-Funktionen sind dazu adjungiert. Sind deshalb die c-Funktion V und die e-Funktion w in einem

Gebiete regulär, so gilt für die Berandung Σ des Gebietes: $\int_\Sigma V d\bar{Z} dw = 0$. Durch Wahl von V mit einer geeigneten Singularität erhält man durch eine Methode, welche der in der Funktionentheorie üblichen ähnlich ist, einen Integralsatz, der dem zweiten Cauchyschen Satze entspricht. Mit seiner Hilfe kann eine reguläre e-Funktion durch die "partie finie" eines Integrals ausgedrückt werden, welches über den im Innern des charakteristischen Kegels des Aufpunktes gelegenen Teil der Fläche Σ zu erstrecken ist, auf welcher die Randwerte vorgegeben sind. Für das Potential p der e-Funktion (welches stets existiert) erhält man eine explicite Integraldarstellung ohne "partie finie." *W. Nef.*

Bureau, Florent. Le problème de Cauchy pour une équation linéaire aux dérivées partielles, totalement hyperbolique d'ordre quatre et à quatre variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 379–402 (1947).

The Cauchy problem for the equation

$$\left\{\frac{\partial^2}{\partial t^2}-a^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)-a^2\frac{\partial^2}{\partial x_3^2}\right\} \times\left\{\frac{\partial^2}{\partial t^2}-c^2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)-a^2\frac{\partial^2}{\partial x_3^2}\right\}u=0,$$

is treated by the well-known (but difficult) method of Hadamard for second order linear equations of normal hyperbolic type. The Cauchy data are assumed to be carried by the hyperplane $t=0$. The Green's formula (formule de réciprocité) shows that the equation is self-adjoint. On account of the constancy of the coefficients, it is comparatively easy to set up an "elementary solution," which is defined explicitly with the help of the function,

$$I(\alpha)=\int_0^\alpha x^{-1}\log(1-x^2)dx=-\tfrac{1}{2}\left\{\frac{\alpha^2}{1^2}+\frac{\alpha^4}{2^2}+\frac{\alpha^6}{3^2}+\cdots\right\}, \qquad 0<\alpha<1.$$

The unknown function u and the elementary solution v are then substituted in the Green's formula which is applied to a certain four-dimensional domain bounded by the hyperplanes $t=0$ and $t=t_0-\eta>0$ and by two suitable hypercones with vertex at $(t_0, x_{10}, x_{20}, x_{30})$. As $\epsilon\to 0$, these hypercones become characteristic surfaces of the differential equation. Green's formula yields a relation consisting of the sum of several terms equated to zero. Each term has a characteristic order of infinity as $\epsilon\to 0$. Hence the sum of all terms of the same order must vanish. In particular, the sum of terms of the form $A(\eta)\log\epsilon$ must vanish. Hence also $\lim_{\eta\to 0}\sum A(\eta)=0$, and this last relation may then be solved for $u(t_0, x_{10}, x_{20}, x_{30})$ in terms of the Cauchy data. The author does not show that the function thus synthesized actually satisfies the differential equation, although this could probably be accomplished. *D. C. Lewis* (College Park, Md.).

Integral Equations

de Cesare, E. A. Elements of the theory of integral equations. Applications to actuarial mathematics and to economic dynamics. An. Soc. Ci. Argentina **144**, 220–274 (1947). (Spanish)

Crum, M. M. On an integral equation of Chandrasekhar. Quart. J. Math., Oxford Ser. **18**, 244–252 (1947).

Chandrasekhar [Bull. Amer. Math. Soc. **53**, 641–711 (1947); these Rev. **9**, 189] has encountered integral equations of the form

$$(*)\qquad 1/H(z)=1-\int_0^1 z(z+t)^{-1}H(t)d\alpha(t), \qquad 0<z<1,$$

where α is given and H is unknown. The author shows that if in (*)

$$\int_0^1|d\alpha(t)|<\infty, \quad \int_0^\delta|d\alpha(t)|\to 0, \qquad \delta\to 0,$$

then the general solution of (*) which is bounded in $0\leqq t\leqq 1$ is given by a contour integral. The relation of (*) to the method of Wiener and Hopf is discussed. *A. Heins.*

Mihlin, S. G. Singular integral equations with continuous coefficients. Doklady Akad. Nauk SSSR (N.S.) **59**, 435–438 (1948). (Russian)

The author studies the equation

$$(1)\qquad a(z)\varphi(z)+b(z)\pi^{-1}\int_L\varphi(\zeta)(\zeta-z)^{-1}d\zeta+T\varphi=f(z);$$

here the integral is in the sense of principal values; L is a "smooth" closed contour of bounded curvature in the complex ζ-plane; z is on L; T is a completely continuous operator in the function-space in which the unknown $\varphi(z)$ is to be found. The author proves for (1) theorems of F. Noether [Math. Ann. **82**, 42–63 (1921)] under the assumptions: (i) $\varphi\in L_2$ (along L), (ii) $a(z)$, $b(z)$ are continuous, (iii) $a^2(z)+b^2(z)\neq 0$ (on L). It is indicated that similar results can be established for systems.

W. J. Trjitzinsky (Urbana, Ill.).

Boruhov, L. The linear integral equation with an almost periodic kernel and a free member. Doklady Akad. Nauk SSSR (N.S.) **57**, 647–649 (1947). (Russian)

The author studies the equation

$$\varphi(x)=f(x)+\lambda L_s[K(x,s)\varphi(s)],$$

where $L_s[\cdots]=\lim\int_0^T[\cdots]ds$ as $T\to\infty$. It is assumed that f is almost periodic in the sense of Bohr $(-\infty<x<+\infty)$ and that $K(x,s)$ is almost periodic in (x,s). It is shown that the Fredholm theory holds. *W. J. Trjitzinsky.*

Vaĭnberg, M. M. On the characteristic values of a class of nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) **58**, 953–956 (1947). (Russian)

The author studies the equation

$$(1)\qquad \lambda u(x)=\int_B K(x,y)g(u(y),y)dy$$

($g(0,x)=0$ for x in the bounded domain B in Euclidean n-space). The following is proved. There exists a sequence $\mu_k\to 0$ such that for every $\lambda=\mu_k$ equation (1) has a real continuous solution, not identically zero, provided: (I) $K(x,y)$ is symmetric, positive (in the sense of integral equations) and its first iterate $K_2(x,y)$ is continuous in x, y; (II) $g(u,x)$ is real and continuous in u, x and $uf(u)\leqq ug(u,x)\leqq a|u|^{p+1}+b|u|$ (real, monotone increasing $f(u)$, $f(0)=0$; a, $b>0$; $0<p<1$). This result connects with some earlier work by the author [C. R. (Doklady) Akad. Sci. URSS (N.S.) **46**, 47–50 (1945); these Rev. **6**, 272], by M. Golomb [Math. Z. **39**, 45–75 (1934)] and by L. Lichtenstein [Vorlesungen über einige Klassen nichtlinearer Differentialgleichungen, Springer, Berlin, 1931]. *W. J. Trjitzinsky* (Urbana, Ill.).

Parodi, Maurice. Remarque sur l'équation intégrale de seconde espèce à noyau singulier de Weyl. C. R. Acad. Sci. Paris **226**, 153–155 (1948).

The equation is $f(t)=g(t)+\lambda\int_0^\infty f(x)\sin xt\,dx$. The author discusses it (formally) by means of the operational calculus and obtains the known results to which Fourier's integral theorem leads. *A. Erdélyi* (Edinburgh).

Parodi, M. Équations intégrales et équations fonctionnelles. Ann. Soc. Sci. Bruxelles. Sér. I. **61**, 199–204 (1947).

Continuation of the work announced in a former note [C. R. Acad. Sci. Paris **222**, 633–635 (1946); these Rev. **9**, 148]. The work is purely formal and uses the familiar technique. Beside integral equations, there appear also integrodifferential equations such as

$$(-)^{n-1}x\int_0^\infty J_0(2(xy)^{\frac{1}{2}})\frac{d^{n-1}}{dy^{n-1}}\frac{\varphi(y)}{y}dy=\varphi(x)$$

which has the solution

$$\varphi(x)=x\int_0^\infty e^{-xt-\frac{1}{2}t^{-1}}F(t^{\lambda}+t^{-\lambda})dt.$$

A. Erdélyi (Edinburgh).

Parodi, Maurice. Équations intégrales et calcul symbolique à deux variables. Revue Sci. **85**, 550–551 (1947).

Nazarov, N. N. On a class of nonlinear integrodifferential equations. Doklady Akad. Nauk SSSR (N.S.) **58**, 741–744 (1947). (Russian)

The author considers the equation

$$(1)\qquad u(x)=\lambda\int_0^1 K(x,y)f[y,u(y),u'(y)]dy.$$

For small λ one has results (I), (II) as follows. (I) If $f(y,u,v)$ is continuous in y and satisfies a Lipschitz condition in u, v, $K_x'(x,y)$ is "piece-wise continuous," while $\int|K(x,y)|dy$, $\int|K_x'|dy<B$, then (1) has a unique continuous solution, approaching 0 with λ. (II) If K_x' is piece-wise continuous, $|K|<N, |K_x'|<N$ (some constant N), $f(y,u,v)$ is a series in nonnegative powers of u, v ($|u|$, $|v|<R$) and is integrable in y, then (1) has a unique analytic solution approaching 0 with λ. It is said that $u(x,\lambda)$ is a "continuous family of solutions" if it satisfies (1) and is continuous in x (on (0, 1)) for all λ on some interval; $u(x,\lambda_0)$ can be "continued" if there exists a continuous family of solutions $u(x,\lambda)$ to which $u(x,\lambda_0)$ belongs; a solution $u(x,\lambda_0)$ is "continuable single-valuedly" if there exists just one continuous family of solutions $u(x,\lambda)$ to which $u(x,\lambda_0)$ belongs; if a solution is not continuable single-valuedly, then $\lambda=\lambda_0$ is its "branch point." The following equation is introduced:

$$(2)\qquad \varphi(x)=\lambda\int_0^1 K(x,y)[A_{00}(y)\varphi(y)+A_{01}(y)\varphi'(y)]dy.$$

If λ_0 is not a characteristic number for (2), then $\lambda=\lambda_0$ is not a branch point of any solution of (1).

W. J. Trjitzinsky (Urbana, Ill.).

Germay, R. H. Sur une méthode d'approximations successives pour l'intégration des systèmes normaux d'équations intégro-différentielles. Ann. Soc. Sci. Bruxelles. Sér. I. **61**, 185–190 (1947).

Let

$$u_{ij}(x)=\int_{x_0}^{x} f_{ij}(x,s,y_k(s))ds,\quad i,j,k=1,\cdots,3,$$

and consider the integrodifferential system

$$(*)\qquad dy_i/dx=F_i(x,y_k,u_{kj}),$$

$y_i=y_i^0$ when $x=x_0$. If the f_{ij} and F_i satisfy certain Lipschitz conditions, it is shown that a solution of (*) can be obtained by a method of successive approximations when $|x-x_0|$ is sufficiently restricted. *F. G. Dressel* (Durham, N. C.).

Monna, A. F. On integral equations for functions with values in a non-Archimedian valued field. Nieuw Arch. Wiskunde (2) **22**, 283–292 (1948).

Let L be a locally compact non-Archimedean valued field, let G be a compact space with countable basis for open sets and let C be the space of all continuous functions from G to L. If $K(x,y)$ is a function of two variables on G with values in L, and if $f(x)\in C$, then also $I(x)=\int_G K(x,y)f(y)dy\in C$, the integral being that previously defined by the author [Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **53**, 385–399 (1944); these Rev. **8**, 33]. The mapping $f(x)\to I(x)$ of C into itself is completely continuous. Using his results on completely continuous operators in non-Archimedean Banach spaces [Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **52**, 654–661 (1943); these Rev. **8**, 33], the author is then able to prove that either the integral equation $f(x)-\int_G K(x,y)f(y)dy=g(x)$ has for all $g(x)\in C$ a unique solution $f(x)$, or else the corresponding homogeneous equation has nontrivial solutions; in the latter case the number of linearly independent solutions is the same as for the transposed equation. *I. S. Cohen.*

Functional Analysis, Ergodic Theory

Dieudonné, Jean. Compléments à trois articles antérieurs. III. Bull. Soc. Math. France **74**, 66–68 (1946).

Steps in proofs of a previous article [same Bull. **72**, 193–239 (1944); these Rev. **7**, 305] are filled in.

J. F. Randolph (Oberlin, Ohio).

Plans Sanz de Bremond, Antonio. Hilbert space of n dimensions. Revista Acad. Ci. Madrid **40**, 59–100, 195–222 (1946). (Spanish)

The paper is an expository account of standard material pertaining to n-dimensional complex vector spaces. The subject matter includes such topics as matrices and linear operators, reducibility, the spectrum and Hermitian matrices.

A. E. Taylor (Los Angeles, Calif.).

***Kramers, H. A. Remarks on the perturbation formulae of Brillouin and Wigner.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 205–210. Interscience Publishers, Inc., New York, 1948. $5.50.

The formula of Brillouin [J. Physique Radium (7) **4**, 1–9 (1933)] referred to in the title consists of a relation which (in case the infinite series contained in it converges) must be satisfied by the eigenvalues F of a Hermitian matrix which differs only little from a diagonal matrix. Wigner [Math. Naturwiss. Anz. Ungar. Akad. Wiss. **53**, 477–482 (1935)] remarked that if the infinite series is broken off after an odd number of terms, the Brillouin formula yields the precise eigenvalues of a certain approximating problem. The present paper consists of a new proof for the formal validity

of the Brillouin formula and of a number of other remarks concerning this formula and Wigner's theorem.

E. H. Rothe (Ann Arbor, Mich.).

Liftšic, I. M. On degenerate regular perturbations. I. Discrete spectrum. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 17, 1017–1025 (1947). (Russian)

Liftšic, I. M. On degenerate regular perturbations. II. Quasicontinuous and continuous spectrum. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 17, 1076–1089 (1947). (Russian)

Let (1) $(L-\lambda)\psi=0$ be a nonperturbed problem, the characteristic values and functions being λ_p, ψ_p (L is a Hermitian operator). The problem is, generally speaking, to find the characteristic values and characteristic functions z_p, φ_p of the perturbed operator $L+t\Lambda$ (Λ, a Hermitian operator; real arbitrary t): (2) $(L+t\Lambda-z_p)\varphi_p=0$. The usual studies have been for $t\Lambda$ "small"; the author is concerned with the contrary case. In a preceding paper [C. R. (Doklady) Akad. Sci. URSS (N.S.) 48, 79–81 (1945); these Rev. 7, 453] he solved (2) when the spectrum of L is discrete and Λ is "regular" in the sense that $\sum|\lambda_p\lambda_q|^{-1}|\Lambda_{pq}|^2<\infty$ ($\Lambda_{pq}=(\Delta\psi_p,\psi_q)$ = matrix element of Λ in the system ψ_p). In part I he studies the case when Λ is "degenerate" in the sense this term has in the theory of integral equations; it is shown that (2) can then be treated on the basis of an algebraic system of a number of linear homogeneous equations. It is indicated how one may achieve the transitions to regular nondegenerate Λ with the "degenerate" theory for the starting point and by a suitable limiting process; the results are of Fredholm type. In part II the author extends the results given in part I to the degenerate cases of quasi-continuous and continuous spectra; these cases are applicable to a number of specific physical problems. *W. J. Trjitzinsky* (Urbana, Ill.).

de Sz. Nagy, Béla. Expansion theorems of Paley-Wiener type. Duke Math. J. 14, 975–978 (1947).

Write $X=\sum a_n x_n$, $Y=\sum a_n y_n$, where the expansions are finite and $\{x_n\}$ is supposed a complete set in H, Hilbert space. The key results are that (I) $\{y_n\}$ is also complete if $\|X-Y\|^2\leqq\lambda_1\|X\|^2+2\mu\|X\|\,\|Y\|+\lambda_2\|Y\|^2$ for some fixed constants λ_1, λ_2, μ satisfying $0\leqq\lambda_1<1$, $\mu^2\leqq(1-\lambda_1)(1-\lambda_2)$; (II) if $\{x_n\}$ is orthonormal then there is a sequence h_n such that $\{(y_n,h_n)\}$ is a biorthogonal sequence with

$$M^{-1}\|u\|\leqq(\sum[u,h_n]^2)^{\frac{1}{2}}\leqq m^{-1}\|u\|,$$

where m, M, $m\leqq M$ are the roots of

$$(1-\lambda_1)-2(1+\mu)x+(1-\lambda_2)x^2.$$

The proofs are compact. Write $TX=Y$. Then $h_n=(T^{-1})^*x_n$. [Actually half of the argument for (I) can be dropped on noting that the author's (7) states a Banach isomorphism so that T^{-1} exists with range H.] *D. G. Bourgin.*

Hartman, Philip, and Wintner, Aurel. On Möbius' inversion. Amer. J. Math. 69, 853–858 (1947).

Consider $y(n)=\sum x(nm)$ and $x(n)=\sum\mu(m)y(nm)$. Refer to these as $y=Ex$ and $x=My$, respectively. The authors show that, with $x\varepsilon l_1$, if $Ex\varepsilon l_1$, then $My=x$ has the unique solution (in l_1) $y=Ex$, while if $Ex\bar{\varepsilon}l_1$, then no solution in l_1 exists. A dual theorem with x and y and My and Ex interchanged is also proved. *D. G. Bourgin* (Urbana, Ill.).

Köthe, Gottfried. Die Quotientenräume eines linearen vollkommenen Raumes. Math. Z. 51, 17–35 (1947).

This paper continues work begun in other papers of the author; the latest of these appeared in Math. Ann. 116, 719–732 (1939). Let μ be a linear space of sequences $x=\{x_i\}$ of complex numbers; then μ^*, its dual space, consists of those sequences $u=\{u_i\}$ such that the scalar product $u\cdot x=\sum u_i x_i$ exists for every x in μ. Assume that μ satisfies the additional condition that $\mu^{**}=\mu$. The earlier papers were largely concerned with solution of linear equations in infinitely many unknowns. The present paper deals with the spaces μ themselves, four natural topologies in them, factor spaces and continuity properties of homomorphisms in terms of the various topologies. *M. M. Day.*

Akilov, G. P. Necessary conditions for the extension of linear operations. Doklady Akad. Nauk SSSR (N.S.) 59, 417–418 (1948). (Russian)

This is a continuation of an earlier note [same Doklady 57, 643–646 (1947); these Rev. 9, 241]. The author notes that the two properties studied in his earlier note are equivalent, but still does not see that they are equivalent to the property which was called in the preceding review P_1; a normed linear space X has property P_1 if and only if for every normed $Y\supseteq X$ there exists a projection with bound one of Y on X. Using this relation the principal results stated are equivalent to the following. No L-space of dimension greater than 2 has property P_1. No n-dimensional normed space ($1<n<\infty$) such that the tangent hyperplane at each point of the unit sphere is unique has property P_1. If X has property P_1 and if T of bound one projects X onto $Y\subseteq X$, then Y has property P_1. *M. M. Day.*

Michal, A. D. Functional analysis in topological group spaces. Math. Mag. 21, 80–90 (1947).

A "bibliographical exposition" written in 1941 with special emphasis on the work of the author and his students. Topics discussed are "functional polynomials and functional power series expansions," "differential calculus of functionals" and "functional differential equations." *C. E. Rickart.*

Rickart, C. E. The singular elements of a Banach algebra. Duke Math. J. 14, 1063–1077 (1947).

Let A be a Banach algebra (normed ring). Let the class of all (left, right) generalized null-divisors be denoted by Z (Z^l, Z^r). A typical definition is $s\varepsilon Z^l$ if $\lim_n sz_n=0$ for a sequence $\{z_n\}$ with $\|z_n\|=1$. The complements in A of Z^l, Z^r, Z will be denoted by H^l, H^r, H. Left, right regular elements and two-sided regular elements are denoted by G^l, G^r, and G. The letters S^l, S^r, S denote classes of singular elements. A principal question which this paper considers concerns the circumstances under which an extension of A is possible in which the elements of H are regularized.

First, general properties of the sets Z, G, S, H are developed. Some of the results are as follows. The class H is the union of the disjoint open sets G and $S\cap H$. Every component of H is either a component of G or consists entirely of singular elements. The notion of a B^* algebra is introduced: to every $x\varepsilon A$ is associated an x^* such that $x^{**}=x$; $(xy)^*=y^*x^*$; $(\lambda x+\mu y)^*=\bar{\lambda}x^*+\bar{\mu}y^*$ for complex numbers λ, μ. The adjoint operation is said to be real provided every self-adjoint element in A has a real spectrum. In this case $S=ZU(Z)^*$. Also, if, in addition, $\sup_x\|x^*\|/\|x\|<\infty$ then $(Z^l)^*=Z^r$ and A is proper (all elements of H are regular). Let A be commutative. Consider the class (A,H) of all

pairs (x, s) with $x \in A$, $s \in H$. Introducing an equivalence relation $(x, s) \sim (x', s')$ in case $xs' = x's$, an algebra of equivalence classes $[A, H]$ is formed which is an extension of A and in which all elements in H are regular. The problem now is to introduce a norm into $[A, H]$ which preserves the norm of A. To this end, the function $r(x, s) = \sup_z \|xz\|/\|sz\|$ is studied. It is proved that for all $x \in A$, $s \in H$, $\lim_k \|x^k\|^{1/k} \leqq n(xs, s) \leqq \|x\|$. Thus if in A, $\|x^2\| = \|x\|^2$, then $n(xs, s) = \|x\|$. Other properties of $n(x, s)$ make it apparent that under these conditions it may be introduced as a norm into the ring $[A, H]$. The paper closes with a discussion of the extension problem for algebras of continuous functions over a bicompact space. *E. R. Lorch* (New York, N. Y.).

Halilov, Z. I. Linear singular equations in normed rings. Doklady Akad. Nauk SSSR (N.S.) **58**, 1613–1616 (1947). (Russian)

Let R be a normed ring of elements $f, g, \cdots$. Let S, operating in R, be (1) distributive; (2) some iterates of the operators ST, TS (T is an arbitrary linear operator of which some iterate is completely continuous) are completely continuous; (3) $S^2 = E$ (identity); (4) some iterate of $(Sf - fS)$ (f in R) is completely continuous; (5) if $(f, Sg) = (S'(f), g)$ for all f, g in R, then S' satisfies (1)–(4). Such an S is termed singular. It is proved that: (I) $Sg(f) = T(f) + gS(f)$, where T is a linear operator of which some iterate is completely continuous; (II) $SgSf = gf + T(f)$, where T is an operator of which some iterate is completely continuous; (III) if $S(f) = g$, then $f = S(g)$. The author studies the equation (1) $K(f) \equiv \varphi f + \psi S(f) + T(f) = g$ (φ, ψ, g given in R; S singular; T linear; some iterate of T is completely continuous). The unknown f is to be in R. It is assumed that $\varphi \pm \psi$ have inverse elements. The kernel K', for which $(g, K(f)) = (K'(g), f)$ (all f, g in R), is termed the adjoint of K; the equation $K'(f) = \chi$ is an adjoint of (1). For the "regular" (in the sense of F. Riesz and Schauder) equation $M(f) \equiv \varphi f + T(f) = g$ and its adjoint $M'(f) = g$ the Fredholm theory holds. The author extends to singular equations (1) an essential part of the results established by N. Noether and N. I. Mushelishvili, and some others, in the field of integral equations. *W. J. Trjitzinsky* (Urbana, Ill.).

Dowker, Yael Naim. Invariant measure and the ergodic theorems. Duke Math. J. **14**, 1051–1061 (1947).

Let m be a countably additive measure function defined on a Borel family F of sets in a space S. Let T be a one-to-one point transformation of S into itself which is measurable in the sense that TA, $T^{-1}A \in F$ for every $A \in F$. A set $A \in F$ is called a null set with respect to T if $m(T^n A) = 0$, $n = 0, \pm 1, \pm 2, \cdots$. The measure m is called potentially invariant with respect to the measurable transformation T if there exists a finite countably additive nonnegative set function μ defined on A for which $\mu(A) = \mu(T^{-1}A)$ and $\mu(A) = 0$ only if A is a null set. Modifications of the ergodic theorems of G. D. Birkhoff and J. von Neumann are readily seen to hold if m is potentially invariant. The principal problem of the present paper is that of the converse proposition. It is shown that, if T is a measurable transformation and if for every $f \in L_1(m)$ the averages $F_n(x) = (1/n)\sum_0^{n-1} f(T^i x)$ converge almost everywhere, then m is potentially invariant. It is also shown that, if T is a measurable transformation with (i) $m(T^{-1}A) = 0$ if $m(A) = 0$, (ii) $f(Tx) \in L_p(m)$ if $f(x) \in L_p(m)$ and (iii) $F_n(x)$ converges in the norm of $L_p(m)$ providing $f(x) \in L_p(m)$, then m is potentially invariant. Also if T is a measurable transformation and if $M_n(A) = (1/n)\sum_0^{n-1} M(T^i A)$ has the property

$$\limsup_{n \to \infty} M_n(A_i) \to 0$$

for every descending sequence $\{A_i\}$ of measurable sets whose intersection is empty, then m is potentially invariant. *N. Dunford* (New Haven, Conn.).

Eberlein, W. F. Abstract ergodic theorems. Proc. Nat. Acad. Sci. U. S. A. **34**, 43–47 (1948).

Let G be a semi-group of linear transformations T of a Banach space E into itself. Let G^* be the set of all transformations of the form $\sum a_j T_j$, $a_j \geqq 0$, $\sum a_j = 1$, $T_j \in G$ and let $O(x)$, the orbit of x, be the set of all points T^*x, $T^* \in G^*$. The semi-group G is called ergodic if it possesses at least one system of almost invariant integrals. By such a system is meant a family (T_α) of linear transformations in E, indexed by a directed set (α), and with the properties (i) $T_\alpha x \in \bar{O}(x)$, (ii) $|T_\alpha| \leqq M$, (iii) $\lim_\alpha (TT_\alpha - T_\alpha)x = \lim_\alpha (T_\alpha T - T_\alpha)x = 0$, $T \in G$, $x \in X$. A proof is indicated for the following theorem. If G is ergodic, x an element of E and (T_α) any system of almost invariant integrals, then the following conditions on an element y in E are equivalent: (i) $y \in \bar{O}(x)$, $Ty = y$, $T \in G$; (ii) $y = \lim_\alpha T_\alpha x$; (iii) $y = \lim_\alpha T_\alpha x$ weakly; (iv) y is a weak cluster point of $T_\alpha x$. An element x is called ergodic if an element $y = T_\alpha x$ exists satisfying any one of these conditions. The ergodic elements form a closed invariant subspace Γ and T_α is a continuous projection in Γ which commutes with every U in G^*. Other results are also stated. One of the chief points of the present treatment is the relaxation of the boundedness and countability restrictions customarily imposed upon G. *N. Dunford* (New Haven, Conn.).

Calculus of Variations

Faedo, Sandro. Su un teorema di esistenza di calcolo delle variazioni e una proposizione generale di calcolo funzionale. Ann. Scuola Norm. Super. Pisa (2) **12** (1943), 121–133 (1947).

The author gives sufficient conditions that a minimum should exist in an unbounded domain when it is known to exist in an arbitrary bounded domain. The integrals considered involve the derivatives of $y(x)$ up to order n. The proofs use a generalization of a known inequality of the theory of differential equations. There is indicated another application of this inequality to obtain the uniqueness of the solution of a nonlinear integral equation. The author points out at the beginning of the paper that the hypotheses in one of his previous papers [Ann. Scuola Norm. Super. Pisa (2) **12**, 1–15 (1943); these Rev. **7**, 525] were inadequate to justify the arguments. The present paper, however, is intended to do more than merely to rectify the error. *L. M. Graves* (Bloomington, Ind.).

***Nöbeling, G. Reguläre Variationsprobleme.** Ber. Math.-Tagung Tübingen 1946, pp. 110–111 (1947).

The present paper is concerned with minima for regular integrals $\int_G F(p, q)dxdy$. Let $\mathfrak{R}$ be a class of curves R in (x, y, z)-space bounding surfaces $z(x, y)$, (x, y) on $G(R)$, and satisfying certain conditions. The paper is concerned with the properties of the subclass $\mathfrak{R}_1$ of $\mathfrak{R}$ for which the minimum problems have solutions. For example, $\mathfrak{R}_1$ is open in $\mathfrak{R}$. No proofs are given. *M. R. Hestenes.*

Pitcher, Everett. A proof of lower semicontinuity. Proc. Nat. Acad. Sci. U. S. A. **33**, 386–392 (1947).

The theorem established in this paper is a generalization of one established by the reviewer, proving semicontinuity of integrals whose integrands are lower semicontinuous functions of their arguments and satisfy a condition closely related to requiring the nonnegativeness of the Weierstrass E-function. The principal interest of the paper resides, however, not in the generalization, but in the method of proof. This proceeds more directly from hypotheses to conclusion than the previously published proof, and avoids the construction of several auxiliary integrands.

E. J. McShane (Charlottesville, Va.).

Wilkins, J. Ernest, Jr. The isoperimetric problem of Bolza with finite side conditions. Bull. Amer. Math. Soc. **53**, 1151–1155 (1947).

It is well known that the problem can be reduced to an equivalent problem without finite side conditions. However, the methods used heretofore introduce certain abnormalities that are undesirable. It is shown that these complications can be avoided by a different but equally simple transformation. Under the transformation it is a simple matter to apply the known results for the problems without finite side conditions so as to obtain their analogues for problems with finite side conditions.

M. R. Hestenes (Los Angeles, Calif.).

Hestenes, Magnus R. An indirect sufficiency proof for the problem of Bolza in nonparametric form. Trans. Amer. Math. Soc. **62**, 509–535 (1947).

Indirect proofs for sufficiency theorems in the problem of Bolza were given by McShane, Myers and the author. In order to apply the author's theorem to the nonparametric problem of Bolza it appears necessary first to transform the problem to a parametric one. The author has remedied this defect and now gives, in the present paper, an indirect proof of the sufficiency theorem without recourse to the transformation. The fundamental theorem of the paper is concerned with a problem in which no differential side conditions appear. The sufficiency theorem for the general problem of Bolza (nonparametric case) then follows immediately from the corresponding result for the associated problem. It is of interest that this associated problem seems to be of a new type and is one to which the usual field theory is not applicable. *H. H. Goldstine* (Princeton, N. J.).

Theory of Probability

Doeblin, W. Sur l'ensemble de puissances d'une loi de probabilité. Ann. Sci. École Norm. Sup. (3) **63**, 317–350 (1947).

This paper appeared previously in Studia Math. **9**, 71–96 (1940); these Rev. **3**, 168. The present version, edited by Fortet, Fréchet, and Loève, contains numerous corrections, minor additions and improvements. *W. Feller.*

Lord, R. D. A problem on random vectors. Philos. Mag. (7) **39**, 66–71 (1948).

The probability density $p_n(x)$ of the x-component X_n of the sum S_n of n independent unit vectors in a plane with uniform distributions for the direction was discussed by Horner [same Mag. (7) **37**, 145–162 (1946); these Rev. **9**, 97]. The author points out that $X_n=\cos\theta_1+\cdots+\cos\theta_n$, where the θ_n are independent rectangular variables; hence the characteristic function of $p_n(x)$ is $J_0^n(t)$. From this Pearson's formula for the density of $|S_n|$ can be derived. These results are generalized to n dimensions.

W. Feller (Ithaca, N. Y.).

Epstein, Benjamin. The mathematical description of certain breakage mechanisms leading to the logarithmico-normal distribution. J. Franklin Inst. **244**, 471–477 (1947).

The mathematical assumptions of the breakage mechanism are essentially the same as introduced by Halmos [Ann. Math. Statistics **15**, 182–189 (1944); these Rev. **6**, 5].

W. Feller (Ithaca, N. Y.).

Epstein, Benjamin. Statistical aspects of fracture problems. J. Appl. Phys. **19**, 140–147 (1948).

An extensive review of the existing literature and a discussion of the applicability of the theory of extreme values to fracture problems. *W. Feller* (Ithaca, N. Y.).

Lüders, Rolf. Eine Verallgemeinerung der Formel von Pollaczek-Geiringer und ihre Anwendung auf die Verteilungsfunktion der Hagelschäden. Z. Angew. Math. Mech. **25/27**, 21–28 (1947).

H. Pollaczek-Geiringer [same Z. **8**, 292–309 (1928)] generalized the familiar time-dependent Poisson process in the following way. Events can occur in singlets, doublets, etc. The n-tuplets are subject to a Poisson process with parameter a_n, and these processes for $n=1, 2, \cdots$ are mutually independent. The author puts in particular $a_n=p(1-p)^{n-2}/n$ for $n\geqq 2$. He suggests that the processes could be periodic in time, so that instead of a_n we would have r parameters $a_{n,1}, \cdots, a_{n,r}$. It is shown how these parameters can be estimated. Two practical illustrations are given but in neither does the assumed periodicity improve the fit of the simpler Geiringer scheme. *W. Feller* (Ithaca, N. Y.).

Kosambi, D. D. The law of large numbers. Math. Student **14**, 14–19 (1946).

Elementary proof of the weak law of large numbers stressing the rôle and the implications of the assumptions.

W. Feller (Ithaca, N. Y.).

Chung, Kai-Lai. On a lemma by Kolmogoroff. Ann. Math. Statistics **19**, 88–91 (1948).

Improvement and generalization of one of the lemmas used by Kolmogoroff in his proof of the law of large numbers [Math. Ann. **102**, 484–488 (1929)]. *M. Loève.*

Parker, J. B. The accumulation of chance effects and the Gaussian frequency distribution. Philos. Mag. (7) **38**, 681–682 (1947).

Continuation of the discussion of how to derive or not to derive part of Laplace's result concerning the sum of n rectangular distributions [given, e.g., in Uspensky, Introduction to Mathematical Probability, McGraw-Hill, New York, 1937, p. 278]. The previous contributions were by Silberstein, Haldane, Goddard, Grimsey [same Mag. (7) **35**, 395–404 (1944); **36**, 184–185, 294–295, 428–433 (1945); these Rev. **6**, 88; **7**, 18, 311]. These papers contain references to numerous more rediscoveries. *W. Feller.*

The reviewer should have stated that for the last result stated in the review the hypothesis $\alpha_i\beta_i \neq 0$ is made, but not the hypothesis $\sum\alpha_i\sum\beta_i \neq 0$

Sarmanov, O. V. Generalization of a limit theorem of the theory of probability to sums of almost independent variables satisfying Lindeberg's condition. Izvestiya Akad. Nauk SSSR. Ser. Mat. 11, 569–575 (1947). (Russian)

Let $\{X_n\}$ be a sequence of random variables. The author considers the conditional probability distribution of X_n for given $X_1, \cdots, X_{n-1}$ and assumes that they obey the Lindeberg conditions uniformly for all possible $X_1, \cdots, X_{n-1}$. The proof of the central limit theorem by means of characteristic functions can then be carried over without difficulty. [For further results of this type cf. Loève, J. Math. Pures Appl. (9) 24, 249–318 (1945); these Rev. 7, 458.]

W. Feller (Ithaca, N. Y.).

Moran, P. A. P. Some theorems on time series. I. Biometrika 34, 281–291 (1947).

Let $X(t)$ be defined by $X(t)=\sum_0^\infty \alpha_i\eta(t-i)$ $(\sum_0^\infty|\alpha_i|<\infty)$, where the η's are mutually independent random variables with a common distribution of zero mean and finite fourth moment. Then, if (*) $\sum_0^\infty \alpha_i \neq 0$, it is proved that $\sum_1^n X(t-s)$ is asymptotically normally distributed as $n\to\infty$. In particular, the theorem is applicable if $X(t)$ defines a stationary stochastic process (wide sense) obtained by solving the difference equation $X(t)+a_1X(t-1)+\cdots+a_kX(t-k)=\eta(t)$, where the roots of the characteristic equation are supposed to have modulus less than 1; (*) is automatically satisfied in this case. Furthermore, if $Y(t)$ is given by $Y(t)=\sum_0^\infty \beta_i\zeta(t-i)$ $(\sum_0^\infty|\beta_i|<\infty,\ \sum_0^\infty\beta_i\neq 0)$ where the ζ's are independent of each other and of the η's, then $\sum_1^n X(t)Y(t)$ is also asymptotically normally distributed as $n\to\infty$. *J. L. Doob.*

Quenouille, M. H. Notes on the calculation of autocorrelations of linear autoregressive schemes. Biometrika 34, 365–367 (1947).

Consider a discrete stationary stochastic process $\{u_n\}$ $(n=\cdots,-1,0,1,\cdots)$ defined by a difference equation $u_{n+j}+a_1u_{n+j-1}+\cdots+a_ju_n=\epsilon_{n+j}$, where ϵ_n is distributed with mean 0 and independently of $\epsilon_{n-1}, \epsilon_{n-2}, \cdots$ and $u_{n-1}, u_{n-2}, \cdots$. Let ρ_i be the "true" correlation between u_k and u_{k+i}, and let r_i be the corresponding sample correlation based on a sample of N observations. The asymptotic covariance between r_k and r_{k+g} is given by $1/(N-k)\sum_{i=-\infty}^{\infty}\rho_i\rho_{i+g}$. The author suggests generating schemes for computing ρ_i and this last expression. A generating function for ρ_i can be obtained by modifying the spectral function to give

$$\sigma^2\sum_{i=-\infty}^{\infty}\rho_i t^i = \lim_{t\to 1}\left[\{1+a_1t+\cdots+a_jt^j\}\left\{1+a_1\frac{c}{t}+\cdots+a_j\left(\frac{c}{t}\right)^j\right\}\right]^{-1},$$

where σ^2 is the ratio of the variance of u_n to that of ϵ_n (the series converges for $a<|t|<1/a$, where a is the largest of the absolute values of the roots of $x^j+a_1x^{j-1}+\cdots+a_j=0$). The above expression is squared to obtain a generating function for $\sum_{i=-\infty}^{\infty}\rho_i\rho_{i+g}$. The computation of the two generating functions is simplified and a numerical example is given. *T. W. Anderson* (Cambridge, England).

Lévy, Paul. Chaînes doubles de Markoff et fonctions aléatoires de deux variables. C. R. Acad. Sci. Paris 226, 53–55 (1948).

Consider a two-dimensional lattice $(m\tau, n\tau)$ and associate with each lattice point a random variable $X_{m,n}$. This system is said to form a two-dimensional Markov chain if for $X_{m,n}$ given along an arbitrary "line" L of lattice points which divides the plane into two regions S and S' the two sets of variables $\{X_{m,n}\}$ associated, respectively, with S and S' are mutually independent. Suppose now that the variables $X_{m,n}$ are normal and stationary so that $E_{p,q}=E(X_{k,k}X_{m,n})$ depends only on the differences $p=|m-k|$, $q=|n-k|$. If $E_{p,q}\leqq E_{0,0}$ the $X_{m,n}$ are a Markov chain if and only if

$$E_{p,q}=\alpha(E_{p-1,q}+E_{p+1,q})+\beta(E_{p,q-1}+E_{p,q+1}) +\gamma(E_{p-1,q-1}+E_{p+1,q+1})+\delta(E_{p-1,q+1}+E_{p+1,q-1})+\theta_{p,q},$$

where $\theta_{p,q}$ equals c or 0 according as $p=q$ or $p\neq q$, and $\alpha, \beta, \gamma, \delta, c$ are constants (which cannot be prescribed arbitrarily). Passing to the limit $\tau\to 0$ one obtains Markov chains associated with all points (x, y). It appears that this process usually leads to a degenerate type: either the correlation between X at different points is too small to assure continuity or the values of X along certain curves completely determine X. These considerations also hold in more than two dimensions. *W. Feller* (Ithaca, N. Y.).

Lévy, Paul. Exemples de processus doubles de Markoff. C. R. Acad. Sci. Paris 226, 307–308 (1948).

Further examples of the limiting process described in the preceding review. It is shown that the two types are not the only ones possible, but all types obtained so far exhibit a certain degeneracy. *W. Feller* (Ithaca, N. Y.).

Sapogov, N. A. The Laplace-Lyapunov limit theorem for singular Markov chains. Doklady Akad. Nauk SSSR (N.S.) 58, 1905–1908 (1947). (Russian)

Let x_n be the variable of a Markov process with possible values $a_1^{(n)}, \cdots, a_{k_n}^{(n)}$, where k_n is finite. It is assumed (1) that $\xi_n=\sum_j a_j^{(n)}/k_n$ and $d_n^2=\sum_j(a_j^{(n)}-\xi_n)^2/k_n$ are finite and $d_n>d>0$; (2) the transition probabilities depend on n but $p_{j,i}^{(n)}>1/(n^\alpha k_n)$ with a constant $0<\alpha<\frac{1}{2}$; (3) one has $E(x_n)=0$ and the higher moments have bounds independent of n. Then the central limit theorem holds for the sums $s_n=x_1+\cdots+x_n$. It is shown in a few lines that this follows from S. Bernstein's now classical proof of the central limit theorem for dependent variables. A similar statement holds if the x_n has a density in the finite interval (a_n, b_n) provided the density of transition probabilities $\phi_n(x, y)$ satisfies an inequality $\phi_n(x, y)\geqq 1/(n^\alpha L_n)$, where $L_n=b_n-a_n>L>0$.

W. Feller (Ithaca, N. Y.).

Fortet, Robert. Sur la probabilité de perte d'un appel téléphonique dans un groupe de x sélecteurs commandés par un orienteur unique. C. R. Acad. Sci. Paris 226, 159–161 (1948).

Suppose that there are k selectors preceded by a unique primary selector C. An incoming call is directed by C to one of the k selectors which remains busy as long as the conversation is going on while C has a dead time of fixed (small) duration τ. An incoming call is lost when all k selectors are busy or if it arrives within time τ after the preceding call (in which case C is busy). To arrive at a Markov process it is assumed that the incoming calls form a Poisson process and that the holding times are exponential. The states are E_j (j selectors busy, C free) or $E_j(x)$ (j selectors busy, C busy for time $x<\tau$). The probability of a lost call is derived from the Kolmogorov equations.

W. Feller (Ithaca, N. Y.).

Pollaczek, Félix. Sur l'application de la théorie des fonctions au calcul de certaines probabilités continues utilisées dans la théorie des réseaux téléphoniques. Ann. Inst. H. Poincaré **10**, 1–55 (1946).

In the study of telephone trunking problems it is usual to introduce assumptions which render the process Markovian so that Kolmogorov's equations can be used. Moreover, it is usual to study only the state of statistical equilibrium. The author became known for a new method of attack [cf. in particular Math. Z. **32**, 64–100, 729–750 (1930); **35**, 230–278 (1932); **38**, 492–537 (1934)]. He goes back to the actual events and the times when they occurred. The situation is described by a point in a certain region of n-space and it is required to evaluate disheartening integrals. This is achieved by a systematic use of complex variables. The theory is more difficult but also considerably more general than the one based on statistical equilibrium. In the present paper the author gives a summary of his work which can be read also by nonspecialists. *W. Feller* (Ithaca, N. Y.).

Arnous, Edmond, Bass, Jean, et Massignon, Daniel. Sur les relations fonctionnelles vérifiées par les fonctions caractéristiques des fonctions aléatoires dérivables en moyenne quadratique. C. R. Acad. Sci. Paris **226**, 627–629 (1948).

The authors show how the functional equations for the joint probability distributions of positions and velocities can be replaced by functional equations for the characteristic functions of the distributions. For example, if in a random motion there are particle accelerations, and if $\gamma(x, u; t)$ is the conditional expectation of acceleration for given position x and velocity u, the authors derive formally the equation $\partial\varphi/\partial t-\lambda\partial\varphi/\partial\mu=i\mu\gamma(-i\partial/\partial\lambda, -i\partial/\partial\mu, t)\varphi$ for the characteristic function $\varphi(\lambda, \mu; t)$ of the joint distribution of x and u at time t. *J. L. Doob* (Urbana, Ill.).

Rice, S. O. Statistical properties of a sine wave plus random noise. Bell System Tech. J. **27**, 109–157 (1948).

Soient Q et q des constantes réelles et I_N une fonction aléatoire réelle du temps t, stationnaire et Laplacienne; l'auteur étudie la fonction aléatoire $I(t)=Q\cos qt+I_N$ (superposition d'un signal sinusoïdal et d'un bruit de fond); en particulier, il détermine: (a) la fonction de répartition de la variable aléatoire I égale à $I(t)$ pour une valeur de t prise au hasard; (b) l'espérance mathématique du nombre de fois que, dans l'intervalle de temps $(0, T)$, $I(t)$ traverse en croissant une valeur donnée, avec l'expression asymptotique de cet espérance mathématique pour $T\to+\infty$; (c) les distributions de l'enveloppe R et de la dérivée $d\theta/dt$ de la phase θ [définies en posant: $I(t)=R\cos(qt+\theta)$]; (d) l'espérance mathématique, asymptotique pour $T\to+\infty$, du nombre de fois que R, ou θ, ou $d\theta/dt$ traversent en croissant des valeurs données, dans l'intervalle de temps $(0, T)$. Ces déterminations sont fournies par formules explicites (faisant fréquemment intervenir des fonctions de Bessel ou hypergéométriques), graphiques, tabulations numériques, etc. Certaines supposent certaines restrictions: spectre de I_N concentré au voisinage de la fréquence $q/(2\pi)$, ou symétrique par rapport à $q/(2\pi)$, ou nul à l'extérieur d'une bande étroite admettant $q/(2\pi)$ pour milieu et uniforme à l'intérieur de cette bande, etc. Ces calculs sont des applications, à but essentiellement pratique, des méthodes exposées par Rice [Bell System Tech. J. **23**, 282–332 (1944); ces Rev. **6**, 89]; l'auteur remarque que certains d'entre eux sont applicables, dans leur principe, à d'autres questions (touchant la modulation de fréquence, par exemple). *R. Fortet* (Caen).

Middleton, David. Some general results in the theory of noise through non-linear devices. Quart. Appl. Math. **5**, 445–498 (1948).

Étant données: une fonction $g(x)$ réelle (caractéristique d'un redresseur ou autre appareil, non linéaire en général), une fonction aléatoire réelle $X(t)$ du temps t, stationnaire et Laplacienne, d'espérance mathématique nulle (bruit de fond), et une fonction certaine (signal modulé en amplitude) $V(t)=A(t)\cos\omega_0 t$ où $A(t)$ est périodique (de période T' différente de $T_0=2\pi/\omega_0$, et en général incommensurable avec T_0), l'auteur étudie la fonction aléatoire $Y(t)=g[X(t)+V(t)]$ (sortie de l'appareil de caractéristique $g(x)$ quand on y place à l'entrée $X(t)+V(t)$). Il définit comme "fonction de correlation" $r(t)$ de $Y(t)$ la quantité:

$$r(t)=\frac{1}{T'T_0}\int_0^{T'}dt'\int_0^{T_0}dt_0 E\{g[A(t')\cos\omega_0 t_0+X(\tau)]\times g[A(t'+t)\cos\omega_0(t_0+t)+X(\tau+t)]\}$$

et en deduit, comme transformée de Fourier de $r(t)$, le "spectre" $W(f)$ de $Y(t)$; une méthode générale est indiquée pour calculer $r(t)$ et $W(f)$, le calcul étant poussé à bout dans quelques cas particuliers (par exemple si $g(x)=\beta(x-b_0)^\nu$ avec $\nu>0$ si $x>b_0$ et $g(x)=0$ si $x<b_0$ (redresseur à loi d'ordre ν), et plus particulièrement si $A=b_0=0$); l'auteur examine aussi le cas des "petits signaux," où la partie utilisée de $g(x)$ est régulière (exclusion de la coupure et de la saturation).

D'autre part, en faisant la restriction que le spectre de $X(t)$ est concentré au voisinage d'une fréquence $f_0=\omega_0/(2\pi)$, l'auteur définit "l'enveloppe" R et la phase θ de $Z(t)=V(t)+X(t)$, en posant: $Z(t)=R\cos(\omega_0 t+\theta)$ [cf. le mémoire analysé ci-dessus, où $A(t)$ est supposé constant], et étudie les distributions de R et θ: distribution du couple $[R(t_1), R(t_2)]$, ou du couple $[\theta(t_1), \theta(t_2)]$, fonction de corrélation de $R(t)$, etc., complètement dans des cas particuliers. *R. Fortet.*

Allard, Georges. Détermination de la valeur la plus probable des grandeurs statistiques. II. La vie moyenne des éléments radioactifs. J. Phys. Radium (8) **8**, 262–269 (1947).

[The first part appeared in the same vol., 212–214 (1947); these Rev. **9**, 295.] The present instalment treats the estimation of the mean lifetimes in a sequence of radioactive transformations using counter data. Special attention is paid to very short and very large means. *W. Feller.*

Vajda, S. The stratified semi-stationary population. Biometrika **34**, 243–254 (1947).

Two populations are subject to mortality dependent on age. The author discusses conditions under which the population sizes can be kept constant by equal numbers of entrants. The result is applied to a population stratified into n grades where members are being promoted into higher grades according to rules depending on age. It is desired to keep all grades constant in size with all new members entering the lowest grade. The time parameter is taken discrete with a finite maximum age. *W. Feller.*

Mathematical Statistics

van Uven, M. J. Extension of Pearson's probability distributions to two variables. I, II. Nederl. Akad. Wetensch., Proc. **50**, 1063–1070, 1252–1264 = Indagationes Math. **9**, 477–484, 578–590 (1947).

In the first paper the author generalizes the differential equation of the Pearson system of probability functions by writing $\partial(\log \varphi)/\partial x = P/G$, $\partial(\log \varphi)/\partial y = Q/H$, where φ is the probability function in two variables, P and Q are linear functions of x and y, G and H quadratic functions of x and y. It is noted that this is not the most general form in two variables, but a special class just as type III is a special case in the one-variable Pearson system. The regression lines are determined and in a certain subgroup they are linear, homoscedastic, and the regression coefficients have the same formulas as in the normal bivariate function. The equations giving the higher moments in terms of the lower ones are indicated. In the second paper the system is classified into six types of which the sixth is the bivariate normal. Two particular cases had been previously considered by K. Pearson [Biometrika **15**, 222–244 (1923)]. Apparently the paper is to be continued as the region of the function is determined only for types I and IIa. *L. A. Aroian.*

Villars, D. S. A significance test and estimation in the case of exponential regression. Ann. Math. Statistics **18**, 596–600 (1947).

Fitting $y = a + be^{-kt}$ with t known and y subject to error is particularly simple when $t_{2i} - t_{2i-1}$ is constant, since the relation of y_{2i} to y_{2i-1} is that of linear regression with equal errors in both variables. Other methods are mentioned, but their comparative efficiency is not discussed.

J. W. Tukey (Princeton, N. J.).

Moran, P. A. P. On the method of paired comparisons. Biometrika **34**, 363–365 (1947).

For the method of paired comparisons, Kendall and Smith [Biometrika **31**, 324–345 (1940); these Rev. **2**, 111] gave without complete proof the first four moments of the distribution of d, the number of triangular circuits (triads) in an oriented complete graph with n vertices, and conjectured its asymptotic normality. In this paper, the author claims to remove the conjectural part of these statements: the moments by what is effectively, though not explicitly, the method of inclusion and exclusion; the normality by the method of moments. The proof is not completely convincing because of omission of crucial details; e.g., for the third moment, triples of triads are distributed into 16 classes without explanation, though the third factorial moment requires distinction only of those triples which belong to a 4-vertex complete graph and those which do not. *J. Riordan.*

***Hald, A. The Decomposition of a Series of Observations Composed of a Trend, a Periodic Movement and a Stochastic Variable.** Thesis, University of Copenhagen, 1948. 134 pp. (English. Danish summary)

The observed dependent variable is assumed to consist of a function of known form of the independent variable (trend), a periodic component (of known period), and a normal component with mean zero and unknown variance σ^2. The different observations are assumed independent, and the unknown regression coefficients enter linearly. These regression coefficients and σ^2 are estimated by standard least squares methods and by the method of moving averages. The usual tests of significance based on the least squares method are given. Application is made to problems in the design of experiments. *J. Wolfowitz.*

Spencer-Smith, J. L. The oscillatory properties of the moving average. Suppl. J. Roy. Statist. Soc. **9**, 104–113 (1947).

The deep difficulties in the problem of removing trends in long time-series are not yet overcome, frequent complications being that (a) low degree polynomials do not work as trend approximations, (b) deviations from trend are serially correlated. The moving average method meets with (a), but generally fails to overcome (b). The author gives numerical illustrations of this difficulty; for instance, if the serial coefficients of the trend deviations are $r_k = (0.7)^k$, a 9 term moving average will not give an efficient smoothing.

H. Wold (Uppsala).

Lindley, D. V. Regression lines and the linear functional relationship. Suppl. J. Roy. Statist. Soc. **9**, 218–244 (1947).

Given that the regression for error-free independent variates upon a dependent variable is linear, a necessary and sufficient condition that the regression still be linear when the independent variates are subject to error is found. This involves the dependent variables only and leads to some interesting interpretations and special cases, some of which are known results. In particular, among others, for a single independent variate there always exist distributions of its errors which preserve linearity, but for multiple regression additional conditions are necessary which are found and shown to be sufficient. A special case of the above is that in which the linear relation between the true values is a functional relation, i.e., the value of the dependent variable is always given by a linear function of the true values of the independent variates. Then either the true or the mean value of the dependent variable is related to the observed values of the independent variates, now subject to error, by a linear regression equation. This situation is discussed in detail in the case of a single independent variate and in particular the estimation of the functional relation from observed data in this case by the method of maximum likelihood is thoroughly explored. It is necessary to assume a definite ratio between the weights to be assigned and then the results are compared with previous work and especially with those obtained by least squares. *C. C. Craig.*

Swineford, Frances. A table for estimating the significance of the difference between correlated percentages. Psychometrika **13**, 23–25 (1948).

Nair, K. R. A note on the mean deviation from the median. Biometrika **34**, 360–362 (1947).

This note shows that the standard error of the mean deviation from the median is equal to or less than the standard error of the mean deviation from the mean for samples of 3 or 4 from a normal population, using some formulas by Walsh [Ann. Math. Statistics **17**, 246–248 (1946); these Rev. **8**, 43]. These results could also have been obtained and extensions made by use of an article on order statistics by Hastings, Mosteller, Tukey and Winsor [Ann. Math. Statistics **18**, 413–426 (1947); these Rev. **9**, 195]. *R. L. Anderson* (Raleigh, N. C.).

Höffding, Wassily. On the distribution of the rank correlation coefficient r when the variates are not independent. Biometrika **34**, 183–196 (1947).

Let (X_i, Y_i), $i=1, \cdots, n$, be identically distributed, independent random vectors with continuous marginal distributions. Let the X_i's be arranged in ascending order. This ordering induces an ordering of the Y_i's; let J be the number of inversions in this sequence. It is proved that if the distribution of (X_i, Y_i) is such that the variance of J is $Kn^3+o(n^3)$, the distribution of $n^{-\frac{3}{2}}(J-EJ)$ approaches a normal distribution as $n\to\infty$. The proof rests on an investigation of the moments. *J. Wolfowitz.*

Daniels, H. E., and Kendall, M. G. The significance of rank correlations where parental correlation exists. Biometrika **34**, 197–208 (1947).

The authors give results on estimation of population rank correlation and on testing whether two samples have significantly different rank correlation coefficients. Let (X, Y) be a chance quantity whose possible values are the N pairs of integers $(1, i_1; 2, i_2; \cdots; N, i_N)$, where $(i_1, \cdots, i_N)$ is a permutation of $(1, 2, \cdots, N)$. Rank correlation between X and Y is expressed as a quantity τ [M. G. Kendall, The Advanced Theory of Statistics, v. 1, Lippincott, Philadelphia, 1944; these Rev. 6, 89]. An unbiased estimator of τ is the rank correlation statistic t, calculated from a random sample of n pairs ($n\leqq N$). Exact and approximate expressions are obtained for σ_t^2. It is shown, under certain conditions on τ and N, that as $n\to\infty$ the distribution of t in standard measure tends to the normal distribution. Examples of estimating τ and testing significance are given. *D. F. Votaw, Jr.* (New Haven, Conn.).

Geary, R. C. Testing for normality. Biometrika **34**, 209–242 (1947).

The author is concerned with the degree to which the results of the classical theory are sensitive to departure from the postulated normality, and finds it is considerable. He exhibits instances where a relatively inconsiderable departure from normality causes the usual tests in the analysis of variance to be seriously vitiated. The efficiencies of several tests of skewness and kurtosis are compared. *J. Wolfowitz* (New York, N. Y.).

Wald, Abraham. A note on regression analysis. Ann. Math. Statistics **18**, 586–589 (1947).

Let $\beta_1, \cdots, \beta_p$ be independent random variables, such that $\beta_1, \cdots, \beta_r$ ($r\leqq p$) are normally distributed with zero means and a common unknown variance σ'^2. For given values of $\beta_1, \cdots, \beta_p$, the variables $y_1, \cdots, y_N$ are assumed to be normally and independently distributed with a common unknown variance σ^2, the mean of y_α being of the form $\beta_1 x_{1\alpha}+\cdots+\beta_p x_{p\alpha}$, where the $x_{i\alpha}$ are constants. It is then shown how confidence limits for the ratio σ'^2/σ^2 may be derived from a set of observed values of the y_α and $x_{i\alpha}$ ($i=1, \cdots, p$; $\alpha=1, \cdots, N$). This constitutes a considerable generalization of earlier results of the author [same Ann. **12**, 346–350 (1941); these Rev. **3**, 174]. *H. Cramér* (Stockholm).

Wald, Abraham. An essentially complete class of admissible decision functions. Ann. Math. Statistics **18**, 549–555 (1947).

Let $x=(x_1, \cdots, x_n)$ and $\theta=(\theta_1, \cdots, \theta_k)$ be points in a sample space M and a parametric space Ω, respectively. The distribution function of x is $F(x, \theta)$, where θ is unknown. Let D be a class of possible decisions d, one of which is to be made on the basis of an observed point x. Usually each element of D will be interpreted as a decision to accept the hypothesis that θ belongs to a certain set ω in Ω. The nonnegative weight function $W(\theta, d)$ expresses the loss caused by taking the decision d when θ is the true parameter point. By a decision function $d=d(x)$, an element d is associated with each sample point x, and $r(\theta, d(x))=\int_M W(\theta, d(x))dF$ represents the corresponding expected value of the loss. If an a priori distribution function $f(\theta)$ of θ is given, a decision function which minimizes the expected value of $r(\theta, d(x))$ corresponding to $f(\theta)$ is called a Bayes solution of the problem of finding a decision function. Continuing his earlier investigations in the field [Ann. of Math. (2) **46**, 265–280 (1945); these Rev. **7**, 21], the author shows that the class C of all Bayes solutions has the following properties: (A) to any decision function $d(x)$, there exists a $d^*(x)$ in C such that (1) $r(\theta, d^*(x))\leqq r(\theta, d(x))$ for all θ; (B) if $d(x)$ is in C, and if $d^*(x)$ is any decision function such that (1) holds for all θ, then the sign of equality must hold for all θ. In the particular case when the decision function is connected with the problem of testing a statistical hypothesis H, this is closely related to previous results by Lehmann [same Ann. **18**, 97–104 (1947); these Rev. **9**, 151]. *H. Cramér* (Stockholm).

Carter, A. H. Approximation to percentage points of the z-distribution. Biometrika **34**, 352–358 (1947).

The author derives three formulas for the approximate percentage points of the z-distribution: (a) one using the exact cumulants of z, (b) the Fisher-Cochran formula [Cochran, Ann. Math. Statistics **11**, 93–95 (1940); these Rev. **1**, 249] with a small correction term added, (c) a new formula similar to (b) but more accurate than (b) at the 20%, 5%, 1%, .1% points if both $n_1, n_2\geqq 20$. The gain in accuracy of (c) over (b) is considerable at the 1% and .1% points. Comparison of (b) and (c) is made over a wide range of n_1, n_2. In the table on page 358, λ at the 5% level should be negative. No comparison is made with Paulson's formula for $F=e^{2z}$ [Ann. Math. Statistics **13**, 233–235 (1942); these Rev. **4**, 23]. The reviewer in tests at the 5% level for values of n_1, n_2; 6, 12; 24, 12; 24, 100; 100, 24; 100, 60; 24, 24; 24, 60; 100, 100, found Paulson's formula slightly superior except for 100, 60, where (c) was better. At the 1% and .1% in the few instances compared, Carter's formula (c) was definitely superior to Paulson's. *L. A. Aroian* (New York, N. Y.).

Krishna Sastry, K. V. On a certain distribution in the theory of sampling. Proc. Nat. Inst. Sci. India **12**, 427–428 (1946).

From a normal probability function with mean zero and standard deviation σ, r samples are drawn with means m_i, variances s_i^2, and individuals n_i, $i=1, \cdots, r$. The distribution of $\xi=\{\sum_{i=1}^r a_i m_i\}\{\sum_{i=1}^r n_i s_i^2\}^{-\frac{1}{2}}$ is derived, where a_i is a constant without restriction. As a special case the author obtains a theorem due to H. Simpson [J. Roy. Statist. Soc. (N.S.) **106**, 266–267 (1943); these Rev. **6**, 9. Cf. also O. Kempthorne, ibid. **107**, 58 (1944); these Rev. **6**, 162]. *L. A. Aroian* (New York, N. Y.).

Armitage, P. A comparison of stratified with unrestricted random sampling from a finite population. Biometrika **34**, 273–280 (1947).

A population π of size N consists of r mutually exclusive subpopulations π_k of size N_k. A variable attached to the

population has mean μ_k and variance σ_k^2 in π_k. It is desired to estimate the quantity $\mu=\sum_{k=1}^{l}N_k\mu_k/N$. Let $\bar{x}$ be the mean of a random sample of n from π. Let $\bar{x}_k$ be the mean of a sample of n_k from π_k with $\sum_{k=1}^{l}n_k=n$. (All sampling is without replacement.) Denote by (a) m_p or (b) m_o the quantity $\sum_{k=1}^{l}N_k\bar{x}_k/N$ when the numbers n_k are selected (a) proportional to N_k or (b) proportional to $\sigma_k\{N_k^2/(N_k-1)\}^{\frac{1}{2}}$, the choice (b) having been shown by Neyman to minimize the sampling variance [J. Roy. Statist. Soc. (N.S.) **97**, 558–625 (1934)]. For infinite populations it is known that (1) $\text{var}\,(m_o)\leqq\text{var}\,(m_p)\leqq\text{var}\,(\bar{x})$. [See S. S. Wilks, Mathematical Statistics, Princeton University Press, 1943; these Rev. **5**, 41.] For finite populations, although the first inequality in (1) of course holds, it is shown that $\text{var}\,(m_p)>\text{var}\,(\bar{x})$ if the μ_k are close enough together. If in addition the quantities $\sigma_k\{N_k/(N_k-1)\}^{\frac{1}{2}}$ are close enough together, $\text{var}\,(m_o)>\text{var}\,(\bar{x})$. It is indicated that (1) holds in most cases of practical importance. Some examples are worked out in detail, illustrated with diagrams. *T. E. Harris.*

Bhattacharyya, A. On some analogues of the amount of information and their use in statistical estimation. II. Sankhyā **8**, 201–218 (1947).

[For part I cf. Sankhyā **8**, 1–14 (1946); these Rev. **8**, 524.] Let $f(x_1, \cdots, x_n; \theta_1, \cdots, \theta_l)$ be an n-dimensional, l-parameter probability density function; $T(x_1, \cdots, x_n)$ an unbiased estimate of the function $\tau(\theta_1, \cdots, \theta_l)$ and $V(T)$ its variance; ν an integer not less than 1; $\iota=(i_1, \cdots, i_l)$, the i_r being nonnegative integers with $\sum_{r=1}^{l}i_r\leqq\nu$ (the condition $\sum i_r\neq 0$ being implicit); ι' defined similarly to ι; $f(\iota)=\partial^{i_1+\cdots+i_l}f/\partial\theta_1^{i_1}\cdots\partial\theta_l^{i_l}$ and similarly for $\tau(\iota)$; $J(\iota, \iota')=\int\{f(\iota)f(\iota')/f\}dx_1\cdots dx_n$; $\|J(\iota, \iota')\|^{-1}=\|J[\iota, \iota']\|$. Under certain regularity conditions it is shown that $V(T)\geqq\sum\tau(\iota)\tau(\iota')J[\iota, \iota']$, the sum being taken over all admissible ι and ι'. This result is invariant under transformation of parameters. The present result generalizes ideas going back to R. A. Fisher [Proc. Cambridge Philos. Soc. **22**, 700–725 (1925)], and discussed by a number of authors. [See H. Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, pp. 477–497; these Rev. **8**, 39.] When $\nu=1$, it is shown that the existence of l minimum-variance unbiased estimates for the l functions $\tau_1, \cdots, \tau_l$ implies that f has the well-known form associated with the existence of a set of sufficient statistics. Nonparametric results are obtained by using difference methods in place of differentiation. Geometric interpretations are given and applications are made to the concept of divergence between two populations, discussed previously by the author [Bull. Calcutta Math. Soc. **35**, 99–109 (1943); these Rev. **6**, 7]. *T. E. Harris* (Santa Monica, Calif.).

Lehmann, E. L., and Scheffé, H. On the problem of similar regions. Proc. Nat. Acad. Sci. U. S. A. **33**, 382–386 (1947).

Assuming that there exists a sufficient statistic for the unknown parameter θ in a certain distribution, Neyman [Philos. Trans. Roy. Soc. London. Ser. A. **236**, 333–380 (1937); cf. also Ann. Math. Statistics **12**, 46–76 (1941); these Rev. **3**, 9] showed that every set in the sample space having a certain structure (the "Neyman structure") is a similar region with respect to θ. In the present note, certain additional assumptions are introduced and, subject to these assumptions, necessary and sufficient conditions are stated for the totality of all similar regions with respect to θ to have the Neyman structure. The proofs are left for a later publication, which will also contain applications to the theory of optimum tests of statistical hypotheses. *H. Cramér* (Stockholm).

***Eisenhart, Churchill, Hastay, Millard W., and Wallis, W. Allen, editors. Selected Techniques of Statistical Analysis for Scientific and Industrial Research and Production and Management Engineering, by the Statistical Research Group, Columbia University.** McGraw-Hill Book Co., Inc., New York and London, 1947. xiv+473 pp. $6.00.

This book discusses a series of problems that occur frequently in planning, analyzing or interpreting quantitative data. Various techniques appropriate to these problems are explained, in terms of both general principles and specific procedures. The book is clearly written, and very complete references are given at the end of each chapter. Although relatively little new statistical theory is included, several topics are given fuller treatment than has previously been available and many new tables are included. The book will be very useful to everybody who has reason to apply the techniques it deals with.

The scope of the book is shown by the headings of its seventeen chapters. (1) Use of variables in acceptance inspection for percent defective. (2) Tolerance limits for normal distributions. (3) Multivariate quality control, illustrated by the air testing of sample bombsights. (4) Effects of rounding or grouping data. (5) Acceptance inspection when lot quality is measured by the range. (6) Variability of amount of inspection for double, multiple, and sequential sampling. (7) Planning and analyzing experiments for comparing two percentages. (8) Planning and interpreting experiments for comparing two standard deviations. (9) Utilization of limited experimental facilities when the cost of each measurement depends on its magnitude. (10) Experiments in which the independent variable is known only in terms of a probability distribution. (11) Planning an experiment for estimating the mean and standard deviation of a normal distribution from observations on the cumulative distribution. (12) Abandoning an experiment prior to completion. (13) Planning experiments seeking maxima. (14) Probability that sample means are in opposite order to population means. (15) Significance of the largest of a set of sample estimates of variance. (16) Inverse sine transformation of proportions. (17) Elements of sequential analysis. *H. Cramér* (Stockholm).

Anscombe, F. J., Godwin, H. J., and Plackett, R. L. Methods of deferred sentencing in testing the fraction defective of a continuous output. Suppl. J. Roy. Statist. Soc. **9**, 198–217 (1947).

The units of the product under test are arranged in the order of production and grouped in lots of suitable size. A deferred sentencing scheme is an inspection method where the decision of accepting or rejecting a particular lot depends not only on the items drawn from this lot but also on the inspection results of some of the immediately preceding and succeeding lots. Thus a final decision on a particular lot is delayed until some of the immediately following lots have been examined. These schemes are nonrectifying and are applicable when the quality changes slowly (gradual trend) except for occasional sudden jumps. One of the functions of such inspection schemes is to give early warning in case of deteriorating quality. Various deferred sentencing schemes are discussed and their merits compared with that of the

sequential probability ratio test when applied to large bulks. Methods for obtaining the operating characteristics of some deferred sentencing schemes are given. *A. Wald.*

Curtiss, John H. Acceptance sampling by variables, with special reference to the case in which quality is measured by average or dispersion. J. Research Nat. Bur. Standards **39**, 271–290 (1947).

The theory and practice of certain types (inferential type, average outgoing quality type) of acceptance sampling plan based on statistical tests of hypotheses are explained. For expository purposes a simple practical example is used based on a problem of manufacturing a small metal cylinder and controlling the volume. The choice of the optimum type of critical region and adjustment of the power of the test is cleared up; further discussion is limited to the parametric case and operating characteristic of the sampling plan. The methods of alternative hypotheses are applied to obtain some general formulas of practical importance for sample sizes and regions of rejection of acceptance sampling plans involving parametric hypotheses. The parameters to which the hypotheses pertain are the mean and the standard deviation. A method is described for dealing with cases where the form of the distribution is nonnormal or not completely specified in advance. The method consists in making use of the various inequalities of the Chebyshev type.

Discussing the case of unknown σ the author is led to recommend that Student's t approach should never be used at all in acceptance sampling work.

Problems of quality measured simultaneously by several parameters are discussed, especially the problem of producer's and of consumer's risk. The producer's risks for individual quality characteristics are always added to get the total producer's risk, but the consumer's risks for each of the quality characteristics must ordinarily be considered separately.

A number of specific formulas for the single sampling case are collected in the appendix. *J. Janko* (Prague).

Solomon, L. Statistical estimation. J. Inst. Actuaries Students' Soc. **7**, 144–173, 213–234 (1948).

Expository paper. *J. Wolfowitz* (New York, N. Y.).

Williams, Kennedy. The classical theory of risk—a statistical approach. J. Inst. Actuaries Students' Soc. **7**, 126–143 (1948).

Mathematical Economics

Caquot, Albert. Stabilité d'un régime économique. C. R. Acad. Sci. Paris **225**, 1101–1104 (1947).

The thesis is advanced that the stability condition of an economic system is given by the relation $d\sum s/\sum s > d\sum p/\sum p$, where, in a unit of time, $\sum s$ is the net disposable income received by the labor force in the form of wages, family allowances, social security payments, etc.; $\sum p$ is the sum of prices of total consumption by labor; and $d\sum p$ is the variation in $\sum p$ with respect to $\sum s$. The composition of the price p of a production good and the relations between the terms are analyzed. Some suggestions as to which terms might be altered to achieve stability are stated.

M. P. Stoltz (Providence, R. I.).

Caquot, Albert. Amélioration du niveau de vie par modification du régime économique. C. R. Acad. Sci. Paris **226**, 529–532 (1948).

This article is an extension of the price analysis referred to in the preceding review. The decomposition of the price of a production good is put in the form:

$$p=\sum_{0}^{n} s_i m_i+\sum_{0}^{n'} s_i' m_i'+\sum_{0}^{n^{IV}} r_i m_i''+\sum_{0}^{n'''} l_i' p_i' \epsilon_i',$$

where s_i and s_i', defined as before, relate to the direct and the lateral operations, respectively; r_i are the state payments to nonactive members of the labor force; m_i, m_i' and m_i'' are the durations, in years, of the operations requiring payments s_i, s_i' and r_i; l_i', p_i' are the savings utilized in each operation and changed with rate of interest ϵ_i'; and i is the index of the operation performed in each category.

It is held that an improvement in national income and hence in the plane of living can result only from appropriate changes in m, m' and m''. The determinants of these quantities are specified and estimates are made, on minimal assumptions, with respect to the improvements in production and real income possible in France in the short period and at the end of five years. *M. P. Stoltz.*

Gallego Diaz, José. Un principe de la moindre action en économie politique. Revue Sci. **85**, 597–600 (1947).

This article consists of translations of the preface to the author's doctoral thesis [University of Madrid] and a supplementary note concerning the application in economics of a principle said to be analogous to the mechanical principle of least action. Given, at time t, a system of n goods, the set of numbers (x_i, y_i, t), $i=1, \cdots, n$, where the x_i are the quantities and the y_i the corresponding prices, is called "economic space." Utility is a function defined on the "economic space" by $\Omega=\phi(x_i, y_i, t)$. Two axioms are assumed. (I) The "economic space" is a Riemann space with quadratic form defined by $ds^2=\psi(x_i, y_i, t)$. (II) (Principle of least economic action.) All trajectories of the economizing individual in the space are the extremals of the integral $\int \Omega ds$, if it exists. These definitions and axioms are intended to serve as the foundation of a mathematical economics. The method is applied to two simple cases of supply and demand curves, i.e., static functions and historical functions.

M. P. Stoltz (Providence, R. I.).

TOPOLOGY

Bernhart, Arthur. Another reducible edge configuration. Amer. J. Math. **70**, 144–146 (1948).

By an argument using Kempe chain considerations, the author proves that, in a regular map, an edge common to two hexagons, abutting two pentagons, is a reducible configuration for four colors. *P. Franklin.*

Kagno, I. N. Corrections. Amer. J. Math. **69**, 872 (1947).

These are six corrections to the author's paper on linear graphs [Amer. J. Math. **68**, 505–520 (1946); these Rev. **8**, 46]. In the list of graphs of degree 6 on page 514, the fifth had been wrongly described; so all the theorems concerning H_6 have to be changed. *H. S. M. Coxeter.*

***Vaidyanathaswamy, R. Treatise on Set Topology. Part I.** Indian Mathematical Society, Madras, 1947. vi+306 pp. Rs. 16–4–0.

This book is the first part of a two volume work based on lectures given by the author at the University of Madras. A feature of the treatment given is its extensive use of partial ordering. A chapter is devoted to the theory of partially ordered sets and lattices. In it various notions are introduced which are used later on for giving lattice theoretic definitions and interpretations of various topological notions. For example, a bicompact space turns out to be a topological space whose lattice of open sets "has a jump at one" and the interiors of closed sets appear as the "normal" elements of this lattice.

There are eleven chapters. The first two contain preliminary material on the algebra of sets and related topics and a third the discussion of partial ordering referred to above. The fourth chapter takes the Kuratowski closure axioms as a point of departure and introduces such notions as open set, closed set, interior, boundary, etc. It concludes with a section each on relative topology in subsets and on metric spaces. Chapter V begins with Hausdorff's neighborhood postulates and after some indications concerning their relationship to the Kuratowski axioms proceeds to discussions of strength of topologies and the various countability and separation axioms. Chapter VI deals with various notions definable in an especially natural way in terms of properties of the lattice of open (or closed) subsets of a topological space and in particular with compactness and bicompactness. In chapter VII functions from one topological space to another are considered. The principal topics treated include continuity, resolution spaces, complete regularity and the extension of continuous functions. Chapter VIII consists primarily of a detailed discussion of the theory of the "localization of hereditary additive properties" of Kuratowski and the related generalization of the notion of derived set. The properties of being finite, of being nondense and of being of the first category are given particular attention. Chapter IX is concerned with finite and infinite direct products of topological spaces. Topologies for the products different from those usually studied are also given attention. A section of the chapter is devoted to connectedness but oddly enough the theorem that the direct product of connected spaces is connected is given only in a very special case. Chapter X deals with the theory of metric spaces with special emphasis on the nontopological notions of uniform continuity, completeness and total boundedness. Chapter XI presents an exposition of a variant, with emphasis on lattice theoretic concepts, of Fréchet's theory of "convergence spaces."

The book contains a very large number of exercises which anyone teaching a course in the subject should find quite useful.

There are quite a few typographical errors. Many of these are corrected in a page and a half list of errata at the end of the book. A nontypographical error is the author's assertion [p. 200] that a direct product of open unit intervals is locally bicompact. It is easily seen that this is the case if and only if the number of factors is finite.

G. W. Mackey (Cambridge, Mass.).

Nachbin, Leopoldo. Sur les espaces topologiques ordonnés. C. R. Acad. Sci. Paris **226**, 381–382 (1948).

Results of the following type are announced. Let X be a (partially) ordered topological space in which (0) "closed" intervals are closed in the topology; (1) there exists an x such that $a, b \leqq x \leqq c, d$ whenever a, b, c, d are given and satisfy $a, b \leqq c, d$; (2) every closed linearly ordered subset which is bounded (in the order) is compact. Then X is a lattice and every subset has a greatest lower and least upper bound. *R. Arens* (Los Angeles, Calif.).

Nachbin, Léopoldo. Sur les espaces uniformisables ordonnés. C. R. Acad. Sci. Paris **226**, 547 (1948).

An ordered space E is called "uniformable" if its topology is based on inverse images of open intervals of real numbers under monotone real valued functions f on E, and if the negation of $x \leqq y$ implies $f(x) > f(y)$ for some f. It is announced that such an E can be embedded in a compact space K with preservation of order and that, if f is a monotone mapping of E into a compact K, then this mapping can be extended to K. *R. Arens* (Los Angeles, Calif.).

Nachbin, Leopoldo. On the axiom of the nonconvergent sequences in some linear topological spaces. Revista Unión Mat. Argentina **12**, 129–150 (1947). (Spanish. French summary)

The topologies considered are based on the notion of convergent series. The discussion is centered around the question of finding conditions under which such topologies satisfy the third axiom [see, e.g., Kuratowski, Topologie I, Warsaw, 1933, p. 77]: if a sequence x_n does not tend to x there exists a subsequence none of whose subsubsequences converges to x. The spaces considered are vector spaces, the topologies those obtained from a sequence of norms where "x_n tends to x" means that for at least one of those norms $\|x_n - x\| \to 0$. In another paper [to appear elsewhere] the author has obtained necessary and sufficient conditions for the validity of this third axiom. In the present paper these results are applied to linear subspaces of the space of sequences $x = (\xi_n)$ with the norm $\|x\| = \sup |\xi_n| \leqq +\infty$. In particular, the space of functions regular in and on the unit circle, the space of entire functions of given maximal exponential type, respectively of given maximal order, the space of functions regular in a fixed closed set are studied. Theorems of the following type are obtained: let B be the space of functions regular in and on the unit circle. If $f_n \to f$ (in B) is interpreted as : there exists a closed circle of radius greater than one in which all f_n and f are regular and the convergence is uniform in this circle, then the topology thus defined in B satisfies the third axiom.

F. Bohnenblust (Pasadena, Calif.).

Vijayaraghavan, T. On two problems relating to linear connected topological spaces. J. Indian Math. Soc. (N.S.) **11**, 28–30 (1947).

[In the original, "linear" was printed as "liner."] The author presents simple examples of a connected linearly ordered topological space which has the power of the continuum but which does not admit an everywhere dense enumerable subset, and of a connected topological space which splits into exactly two connected open and closed subspaces when any one point is removed but is not a linearly ordered space. *G. W. Mackey.*

Sreenivasan, T. K. Some properties of distance functions. J. Indian Math. Soc. (N.S.) **11**, 38–43 (1947).

Let X be a space with metric $d(x, y)$. Let $f(z)$ be an ordinary function with $f(0) = 0$, $f(z) \geqq 0$ when $z \geqq 0$. Then a necessary and sufficient condition is found that

$d'(x, y) = f(d(x, y))$ be also a metric function, the condition being $f(z) \leqq f(u)+f(v)$ when $0 \leqq v-u \leqq z \leqq u+v$. It is shown that this condition is fulfilled when f is nonconstant, increasing and concave. It is pointed out that the topology of X need not be preserved. Transformations of countable families of pseudometrics are also considered.

R. Arens (Los Angeles, Calif.).

Keldyš, L. V. Continuous images of a zero-dimensional compact set. Doklady Akad. Nauk SSSR (N.S.) **58**, 1585–1588 (1947). (Russian)

The author extends a result of Vaĭnšteĭn [same Doklady (N.S.) **57**, 431–434 (1947); these Rev. **9**, 153]. A continuous mapping f of a zero-dimensional compact metric space X onto an n-dimensional (compact) metric space Y is said to have a representation A if there exist a space Z and continuous maps $\psi: X \to Z$ and $\varphi: Z \to Y$ such that $f = \varphi\psi$, Z has dimension zero, and $|\varphi^{-1}(y)| < \aleph_0$ for every $y \in Y$. A dissection of X is a division of X into a (finite) number of disjoint open and closed sets. A sequence of dissections such that every dissection is a subdissection of its predecessor is called a descending sequence of dissections. The following theorem is stated. A continuous map $f: X \to Y$ has a representation A, $f = \varphi\psi$, where $|\varphi^{-1}(y)| \leqq m$ for every $y \in Y$, if and only if for every monotone decreasing sequence $\{\epsilon_k\}_{k=1}^{\infty}$ of real numbers with limit 0, there exists a descending sequence of dissections of X such that the kth dissection induces a covering of Y, each element of which has diameter at most ϵ_k, and such that the intersection of any m elements of the covering of Y is void. Two closely related theorems are stated, and an example is constructed of a continuous map of the Cantor set onto the unit square which admits no representation A.

E. Hewitt (Chicago, Ill.).

Hu, Sze-tsen. A new generalization of Borsuk's theory of retracts. Nederl. Akad. Wetensch., Proc. **50**, 1051–1055 = Indagationes Math. **9**, 465–469 (1947).

Borsuk's theory of absolute retracts and absolute neighborhood retracts is given an obvious generalization to compact (= bicompact) Hausdorff spaces.

S. Eilenberg.

Cartan, Henri. Sur la cohomologie des espaces où opère un groupe. Notions algébriques préliminaires. C. R. Acad. Sci. Paris **226**, 148–150 (1948).

Cartan, Henri. Sur la cohomologie des espaces où opère un groupe: étude d'un anneau différentiel où opère un groupe. C. R. Acad. Sci. Paris **226**, 303–305 (1948).

In the first note definitions are made which supplement those of Koszul [same C. R. **225**, 217–219 (1947); these Rev. **9**, 196]. A ring A is called doubly graded if A is given as a direct sum of subgroups $A^{p,q}$ (p and q nonnegative integers) such that $A^{p,q} \cdot A^{r,s} \subset A^{p+r,q+s}$ and $A^{p,q} \cdot A^{p,r} \subset A^{p,q+r}$. Define $A^n = \sum_{p+q=n} A^{p,q}$ and $w(x) = (-1)^n x$ for $x \in A^n$. Suppose further that two additive operators d_1, d_2 are given such that $d_1(A^{p,q}) \subset A^{p+1,q}$, $d_2(A^{p,q}) \subset A^{p,q+1}$, $d_1(xy) = (d_1x)y + w(x)d_1(y)$, $d_2(xy) = (d_2x)y + w(x)d_2(y)$, $d_1d_1 = d_2d_2 = 0$, $d_1d_2 + d_2d_1 = 0$, $d_1w + wd_2 = d_2w + wd_2 = 0$. Then $d = d_1 + d_2$ is a differentiation operator in A and the sequence of Leray-Koszul [loc. cit.] may be defined.

In the second note the case is considered when a group G operates on a singly graded ring A with a differentiation operator. The "cochains" of G with values in A then form a doubly graded ring as above and the Leray-Koszul sequence of this ring gives invariants of the ring A with G as a group of operators. If E is a locally compact space, union of a sequence of compact sets, and if the group G operates on E discretely and without fixed points, the above considerations lead to invariants of E as a space with operators and to connections with the invariants of the quotient space E'.

S. Eilenberg (New York, N. Y.).

Lichnerowicz, André. Sur les nombres de Betti de certaines variétés différentielles définies comme espaces fibrés. C. R. Acad. Sci. Paris **226**, 212–214 (1948).

Let B and E be compact orientable differentiable manifolds of dimensions n and $n+q$, respectively, and let E be a fibre space over B, with fibres of dimension q. The main aim of this paper is to generalize the results of a previous note [same C. R. **224**, 1413–1414 (1947); these Rev. **9**, 102] concerning the relations between the Betti numbers of E and B. The author introduces a differential form Θ of degree q in E, which reduces locally to a monomial $\Theta = A d\xi^1 \cdots d\xi^q$, where the system $d\xi^\alpha = 0$ defines a field transversal to the fibres. An operation is introduced which maps differential forms in E to those of the same degree in B. It is then stated that this operation induces a homomorphism of the cohomology groups $H_i(E)$ onto the cohomology groups $H_i(B)$, if the form Θ is closed. As a corollary it follows that for a certain class of Finsler spaces the fibre space of unit tangent vectors collinear to their elements of support has for its Betti numbers those of the topological product of B and the $(n-1)$-dimensional sphere S_{n-1}.

S. Chern (Nanking).

Roberts, J. H. A problem in dimension theory. Amer. J. Math. **70**, 126–128 (1948).

If X is a compact metric space of dimension n then it is known that in the space I^X_{2n+1} (metrized by the usual least upper bound metric) of continuous mappings of X in the $(2n+1)$-cube I_{2n+1}, the homeomorphisms of X in I_{2n+1} form a dense G_δ-set. If X is only separable metric, I^X_{2n+1} is known to contain a dense G_δ-set of homeomorphisms, and the question was asked [Hurewicz and Wallman, "Dimension Theory," Princeton University Press, 1941, in particular, footnote on p. 60; these Rev. **3**, 312] whether it is true that even in this case the full set of homeomorphisms constitutes a G_δ-set. The present paper answers this question in the negative. It is proved that, if $n = 0$ and A is a subset of the Cantor set C for which $C - A$ is dense in C, then there exists an n-dimensional set X with the following property. In the space of continuous mappings of X in I_{n+1} there is a set T, homeomorphic to C under a homeomorphism g, such that if $t \in T$ and $g(t) = x$ for $x \in C$, then t is a homeomorphism of X in I_{n+1} if and only if $x \in A$. The negative answer to the question above then follows upon taking A as any non-G_δ subset of C and noting that I_{n+1} is a subset of I_{2n+1}.

H. Wallman (Cambridge, Mass.).

Mychkis, A. Sur un lemme géométrique qui s'applique dans la théorie de stabilité au sens de Liapounoff. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 295–298 (1947).

Let G be a domain in the $(n+1)$-dimensional space of the real variables $x_1, \cdots, x_n, t$. Let S be an $(n-1)$-dimensional closed manifold in the subspace $t = 0$, such that the origin of coordinates is contained inside S. Consider a family of (mutually disjoint) trajectories $T(A, t)$ $(0 \leqq t < \infty, A \in \overline{S \cap G})$, where $T(A, t) \in E^t_{n+1}$, $T(A, 0) = A$, $T(A, t) \neq (0, \cdots, 0, t)$ (E^t_{n+1} denotes the subspace of E_{n+1} corresponding to a fixed value of t). It is assumed that $T(A, t)$ is a continuous function in the arguments A, t. The author states conditions on the domain G and the function $T(A, t)$ which guarantee the existence of a point $A \in \overline{S \cap G}$, such that $T(A, t) \in \bar{G}$ for all

$t \in [0, \infty)$. It seems to the reviewer that, contrary to the statement by the author, the continuity of $T(A, t)$ should be required only for pairs (A, t) such that $T(A, t) \subset \bar{G}$. Should continuity be assumed for all (A, t) the conditions imposed on the function $T(A, t)$ would trivially imply $T(A, T) \subset \bar{G}$ for all pairs (A, T). *W. Hurewicz* (Cambridge, Mass.).

Cuesta, N. On the concept of line. Revista Mat. Hisp.-Amer. (4) 7, 249–254 (1947). (Spanish)

After giving several simple examples the author concludes that "line" should be defined as a one-dimensional set which is the continuous image of a rectilinear line.

J. H. Roberts (Durham, N. C.).

Bing, R. H. Solution of a problem of R. L. Wilder. Amer. J. Math. 70, 95–98 (1948).

Wilder's question [Amer. J. Math. 53, 39–55 (1931)] was as follows. Suppose M is a connected im Kleinen set which is the sum of two sets M_1 and M_2 such that (1) each is irreducibly connected from a point A to a point B, and (2) $M_1 \cdot M_2 = A + B$. Then is M necessarily a simple closed curve? The author answers the question in the negative. He gives not merely one example, but a collection G of point sets filling a square plus its interior, each irreducibly connected between the same two points A and B, such that the sum of any two of these can be taken as M. Strong use is made of the well-ordering theorem. The following modification of Wilder's question remains unanswered: if, in addition to the hypotheses above, M is a G_δ set, then is M a simple closed curve? [Wilder showed that if M is a locally compact continuum then M is a simple closed curve.]

J. H. Roberts (Durham, N. C.).

GEOMETRY

Haantjes, J. Equilateral point-sets in elliptic two- and three-dimensional spaces. Nieuw Arch. Wiskunde (2) 22, 355–362 (1948).

A point set in a metric space is called equilateral if its pairs of different points have the same distance. A method is given for deriving all equilateral n-tuples in elliptic space. In particular, in the elliptic plane there are equilateral sextuples, but no septuples. In three-dimensional space there are six kinds of (noncongruent) equilateral quintuples, two kinds of sextuples and no septuples. *H. Busemann.*

Bottema, O. Surfaces with Euclidean metric in non-Euclidean geometry. Nieuw Arch. Wiskunde (2) 22, 246–268 (1948). (Dutch)

This is mainly a historical exposition of the classical problem suggested by the title, with emphasis on the elliptic case. Bianchi [Ann. Mat. Pura Appl. (2) 24, 93–129 (1896)] proved that the binormals of any curve of unit torsion in elliptic space are Clifford-parallel and generate a surface of absolute curvature zero. Whitehead [Proc. London Math. Soc. (1) 29, 275–324 (1898)] cited a certain sextic surface as a "simple" instance. This shows that, as late as 1898, such distinguished Englishmen as Whitehead and Burnside were unaware of the existence of the Clifford surface, which was already well known in Germany because Klein had learnt of it from Clifford himself by attending a meeting of the British Association in 1873. The author describes another interesting example: the quartic surface

$$(x_1x_4 - x_2x_3)^2 + (x_1^2 + x_4^2)^2 = (x_1x_3 + x_2x_4)^2 \tan^2 r,$$

which is generated by Clifford-parallel lines intersecting a fixed circle of radius r. *H. S. M. Coxeter.*

Cassina, U. Nuova teoria della congruenza geometrica. Period. Mat. (4) 25, 196–213 (1947).

The paper presents a new axiomatization of congruence in terms of eight postulates involving the three primitive notions: point, segment and congruence of two pairs of distinct points. The ordered independence of the postulate set is established (that is, the first postulate is shown independent of the remaining seven, while each of the other postulates is proved independent of all those that precede it), and most of the theory of congruence given by Hilbert [Grundlagen der Geometrie, 7th ed., 1930, pp. 11–28] is derived from the author's formulation. A feature of the new development is the elimination of "congruence of angles" as a primitive notion. *L. M. Blumenthal.*

Schützenberger, Marco. Axiomatisation de la géométrie dans un complexe linéaire de droites. Revue Sci. 85, 782–784 (1947).

The author suggests the following framework for the axiomatic treatment of linear complexes. There are given two kinds of basic elements, points and lines, and an incidence relation between them which satisfies the following postulates. (A′) Two distinct points are on at most one line. (A″) Two distinct lines have at most one point in common. (Such a system has often been called an "incomplete projective plane.") (B) If P is a point and h is a line, and if P is not on h, then there exist one and only one point P', one and only one line h', such that h and h' meet in P and such that h' connects P and P'. Every linear complex meets these requirements; but there exist configurations with properties (A) and (B) which cannot be realized as linear complexes. The author therefore adds the following postulate (C) if the points $P(i)$ for $i = 1, \cdots, 5$ form a pentagon so that $P(i)$ and $P(i+1)$ are distinct points connected by a line, if the point $Q(i)$ is the uniquely determined point on the line from $P(i+2)$ to $P(i+3)$ which is connected with $P(i)$ by a line, then there exists a point $R(i)$ on the line from $P(i)$ to $Q(i)$ which may be connected by lines with $Q(i-1)$ and $Q(i+1)$. This postulate is likewise satisfied by every linear complex; and the author sketches a proof of the following partial converse. If a configuration satisfies postulates (A) to (C), and if it is derived from some pentagon by connection of points and intersection of lines and application of postulate (B), then it is a linear complex in a projective space over a prime field (of characteristic 0 or p).

R. Baer (Urbana, Ill.).

***Kerékjártó, Béla. A Geometria Alapjairól. Második Kötet. Projektív Geometria. [The Foundations of Geometry. Volume Two. Projective Geometry].** Magyar Tudományos Akadémia, Budapest, 1944. xxix+613 pp.

This is the second volume of a treatise on the foundations of geometry; the preceding volume dealt with Euclidean geometry. The classical projective geometry is developed in great detail so that this book can also be used as a text book.

The author's aim is to give a foundation of projective geometry on which it is possible to build either Euclidean, hyperbolic or elliptic geometry. The greater part of the book is confined to the discussion of real projective geometry. The author avoids imaginary elements because, on the chosen axiomatic basis, their use could only mean a change in terminology. An analytical discussion of complex projective geometry is given separately.

In chapter I the axioms of projective geometry are introduced. They are arranged in the following three groups: axioms of connection, axioms of order and an axiom of continuity which is appropriately called Dedekind's axiom. The axioms of order for the Euclidean geometry are modified to permit cyclical ordering of the one-dimensional primitive forms. In a preliminary discussion all axioms are presented as theorems derived from the axioms of the Euclidean geometry as given in the first volume. Chapter II discusses the projective geometry of the straight line. Projectivities between pencils of points and groups of projective correspondences on a line are investigated. Involutions and commutative correspondences are studied in detail. Nets of rationality are defined and used to introduce projective coordinates. Chapter III deals with the projective geometry of the plane. First the topological structure of the projective plane is discussed. Projective correspondences between different planes as well as correspondences between points of the same plane are studied and classified. Correlations and polarities are defined and are thoroughly analyzed. Correlations commutative with polarities are studied. In chapter IV the projective geometry of three-dimensional space is treated in a similar manner. Collineations, correlations and polarities are classified. Collineations of the space commutative with a given polarity in an invariant plane as well as collineations commutative with a given polarity of the space are studied extensively. Finally a brief analytical discussion is given. Chapter V gives an account of the properties of conics; Chapter VI deals with the surfaces of second order. The classical theorems are developed and the group-theoretical viewpoint is stressed. The affine and metric properties of conics and quadric surfaces are briefly mentioned. Chapters V and VI, like the preceding chapters, are concluded by an analytical discussion.

Chapter VII introduces a metric. The projective plane is treated here in great detail, the space only briefly. Certain subgroups of the group of collineations lead either to the elliptic, hyperbolic or parabolic (Euclidean) geometry. These subgroups are characterized and a possible axiomatic treatment of the three geometries is indicated. The author constructs models for these geometries and offers an analytical discussion. The analytical treatment is made more uniform by introducing complex numbers. This leads easily to a concise investigation of imaginary elements and of complex projective geometry. Chapter VIII analyzes the axioms of projective geometry. The system of axioms proposed by Bieberbach [Einleitung in die höhere Geometrie, Teubner, Leipzig, 1933] and by Veblen and Young [Projective Geometry, Ginn, New York, 1910, 1918] are studied. The question of independence of the axioms is discussed here. In this connection it should be remarked that the axioms used by the author are not independent. The author justifies this by didactical arguments. Finite projective geometries as well as the projective geometry of of the n-dimensional space are just mentioned. In the second half of this chapter the rôle of the theorems of Desargues and Pappus as well as of the axiom of Archimedes are scrutinized. The connection between projective geometry and number fields is discussed.

Certain rather classical topics of projective geometry are not covered by the text. For instance the geometry of lines is not treated at all; the null-system is just mentioned as a particular correlation. Curves and surfaces of higher than the second order as well as nonlinear correspondences are not discussed. The omission of these topics can be justified for various reasons. Some of these topics could hardly be fitted in a natural way into the axiomatic setup of the book; others would require an unduly large amount of space.

E. Lukacs (Cincinnati, Ohio).

Suseela, M. Non Desarguesian geometry. Math. Student **14**, 1–13 (1946).

In this paper a survey is given of the various possibilities of introducing coordinates in a projective plane which arise by considering special cases of the postulate of Desargues on the one hand and generalizations of the concept of field on the other hand. The generalized fields discussed are, apart from some new intermediate types: nondistributive fields [Dickson], alternating fields [Cayley-Zorn-Moufang], Cartesian number systems [Baer]; each of these types, as well as the intermediate ones, is shown to be equivalent to a certain special case of the postulate of Desargues. The method used is essentially one developed by Baer [see Amer. J. Math. **64**, 137–152 (1942); also Trans. Amer. Math. Soc. **56**, 94–129 (1944), in particular, p. 104, lemma 3; these Rev. **3**, 179; **6**, 98]. The author did not discuss the still more general method of introducing coordinates which is due to M. Hall [Trans. Amer. Math. Soc. **54**, 229–277 (1943); these Rev. **5**, 72]. *R. Baer* (Urbana, Ill.).

Hall, Marshall, Jr. Cyclic projective planes. Duke Math. J. **14**, 1079–1090 (1947).

Eine endliche ebene projektive Verknüpfungsgeometrie mit einer cyclischen transitiven Gruppe von Collineationen gestattet eine analytische Darstellung mit Hilfe einer erzeugenden Folge ganzer Zahlen $a_0, \cdots, a_n$ (mod N), bezeichnet als "difference set," so dass für jedes ganze $d \not\equiv 0$ (mod N) die Kongruenz $a_i - a_j \equiv d$ genau 1 Lösungspaar a_i, a_j aus der Folge besitzt. Ein Punkt der Geometrie ist gegeben durch eine ganze Zahl k (mod N), eine Gerade g_l durch die auf ihr liegenden Punkte, nämlich die Zahlen (mod N) $a_0+l, a_1+l, \cdots, a_n+l, l$ beliebig ganz. Die Abbildung $i \to i+1$ ist eine Collineation, die Zuordnung des Punktes m zur Geraden g_{-m} eine Polarcorrelation. Entsprechend lässt sich eine unendliche ebene Geometrie mit cyclischer Gruppe mit Hilfe einer unendlichen Folge $\{a_i\}$ definieren; die Geometrie ist dann nicht notwendig Desargues'sch, sie liefert jedoch kein Modell für ein freies projektives Netz ohne Configurationen. Die genauere Untersuchung der endlichen Geometrieen gründet sich auf arithmetische Eigenschaften von n und $N=n^2+n+1$, und auf die Existenz von weiteren Collineationen ausser den cyclischen. Es ist zu vermuten, dass nur zu einem n, das Primzahl oder Primzahlpotenz ist, eine endliche projektive Ebene mit cyclischer transitiver Gruppe existiert, und dass dann stets der Desargues'sche Satz gilt. *R. Moufang* (Frankfurt am Main).

Benneton, Gaston. Configurations harmoniques et quaternions. Ann. Sci. École Norm. Sup. (3) **64**, 1–58 (1947).

Most of the important self-dual configurations of points and planes in complex projective 3-space are self-polar for

at least one quadric. Consequently a suitable choice of coordinate system enables us to represent a point and its corresponding plane by the same set of four coordinates, and then to represent both the point (x, y, z, t) and the plane $[x, y, z, t]$ by the quaternion $\xi = x+yi+zj+tk$, so that the incidence condition for point ξ and plane η (or vice versa) is $S\xi\eta = 0$. In terms of arbitrary quaternions P and Q, the collineations and correlations that preserve the fundamental quadric $N\xi = 0$ are represented by the transformations $\xi \rightarrow P\xi Q$ and $\xi \rightarrow P\bar{\xi}Q$ (not respectively), or let us say briefly the transformations $P\xi Q$ and $P\bar{\xi}Q$.

In terms of pure quaternions (or vectors) U and V, the involutory collineations are as follows: $U\xi V$ is a harmonic biaxial homology whose two axes are polar lines (the "half-turn" of elliptic geometry), $U\xi$ and ξV are harmonic biaxial homologies whose axes are generators of the fundamental quadric (i.e., Clifford translations of length $\frac{1}{2}\pi$), and $P\bar{\xi}P$ is a harmonic central homology (or reflection in the plane P). Letting J or J' denote any one of $1, i, j, k$, we obtain an Abelian G_{16} of collineations $J\xi J'$, and an Abelian G_8 of collineations $J\xi J$ and $J\bar{\xi}J$. These are subgroups of a G_{192} consisting of $K\xi\bar{K}J$ and $K\bar{\xi}\bar{K}J$, where K is any one of the 24 quaternions J, $J \pm J'$, $1 \pm i \pm j \pm k$. This group, which leaves the tetrahedron of reference invariant, occurs as a subgroup in the direct product of the two G_{24}'s $K\xi$ and ξK. [The otherwise excellent bibliography fails to mention the highly relevant paper by Goursat, same Ann. (3) 6, 9–102 (1889).] When applied to a point of general position (and its polar plane), the G_{16} yields Kummer's configuration 16_6, the G_8 yields a pair of desmic tetrahedra (and their polars), and either of the G_{24}'s yields a complete desmic configuration 24_{8+3}. Moreover, the G_{16} leaves invariant the fifteen tetrahedra of Klein's 60_{15}.

When regarded as correlations, $U\bar{\xi}$ and $\bar{\xi}V$ are null polarities; e.g., any set of six mutually commutative null polarities may be expressed as $I\bar{\xi}$ and $\bar{\xi}I$, where I stands for i or j or k. On the other hand, $U\bar{\xi}V$ and $P\bar{\xi}P$ are symmetric polarities, and it is convenient to use the same symbols for the corresponding quadrics. The nine quadrics $I\bar{\xi}I'$ form with the fundamental quadric a set of ten which are all preserved by the G_{16}; and the eight quadrics $J\bar{\xi}J$ and $J\xi J$, or $x^2 \pm y^2 \pm z^2 \pm t^2 = 0$, are all preserved by the G_8. Every set of mutually commutative symmetric polarities determine quadrics belonging to such a set of eight or ten.

In his final section, the author proves and generalizes the results stated without proof in his earlier papers [Bull. Sci. Math. (2) 68, 190–192 (1944); C. R. Acad. Sci. Paris 220, 548–550, 640–642 (1945); these Rev. 7, 69, 166].

H. S. M. Coxeter (Toronto, Ont.).

Grosheide F. Wzn, G. H. A. On the figure of four projective spaces $[n_1-1]$, $[n_2-1]$, $[n_3-1]$ and $[n_4-1]$ in a $[n-1]$, where $n_1+n_2+n_3+n_4=2n$. I, II. Nederl. Akad. Wetensch., Proc. 50, 1126–1133, 1302–1309 = Indagationes Math. 9, 540–547, 628–635 (1947).

An integrity basis in the general case (no mutual dual spaces, $n_1 \leqq n_2 \leqq n_3 \leqq n_4$) is shown to consist of

$$I_p = (d^{n_4} a^p b^{n-n_4-p})(c^{n_3} a^{n_2-p} b^{n-n_3-n_4+p}), \qquad p = 0, 1, \cdots, T,$$

where $T = n_1$ if $n_1 \leqq n-n_4$ and $T = n-n_4$ if $n_1 \geqq n-n_4$. In cases in which dual spaces occur some invariants of the type $(\alpha^{n_4}\beta^{n_4})$ must be added to this system. The author analyses the different possibilities and discusses the smallest integrity basis, using the T line-transversals of the four spaces in the general case. An equation for the crossratios S of the four intersection points on these transversals is derived. The concomitants to be used in a projective classification of sets of four spaces in which the S-equation has one or more multiple roots are given.

E. M. Bruins (Amsterdam).

Srb, Jan. Sur les simplexes autopolaires d'une polarité de l'espace à n dimensions. Časopis Pěst. Mat. Fys. 72, 49–59 (1947). (Czech. French summary)

Deux simplexes différents autopolaires sont de la même espèce quand il existe une correspondance biunivoque telle qu'à chaque arête de l'un des deux simplexes correspond une seule arête de l'autre de sorte qu'aux arêtes passant par un sommet de l'un des simplexes correspondent les arêtes passant par un sommet de l'autre et que les involutions des pôles harmoniques engendrées sur les arêtes correspondantes soient de la même espèce. L'article contient le théorème suivant. Les simplexes autopolaires d'une même polarité de l'espace à n dimensions sont de la même espèce. A l'aide de ce théorème l'auteur effectue la classification projective des polarités et aussi celle des quadriques de l'espace à n dimensions par rapport aux homographies réelles.

F. Vyčichlo (Prague).

Villa, Mario. Sulle trasformazioni puntuali fra due spazi ordinari in una coppia a Jacobiano nullo. Boll. Un. Mat. Ital. (3) 2, 95–103 (1947).

Gambier, Bertrand. Cercles focaux d'une conique. J. Math. Pures Appl. (9) 25 (1946), 241–255 (1947).

Les cercles bitangents à une conique à centre (cercles focaux) forment 2 familles, qui correspondent respectivement, par conservation de chaque centre et multiplication du rayon par un rapport constant (réel ou imaginaire), aux cercles des deux faisceaux linéaires ponctuels, orthogonaux entre eux, admettant les foyers de la conique pour points de base ou pour points limites [les carrés des rayons des cercles focaux sont fonctions du deuxième degré de l'abscisse de leur centre]. Toutes les coniques admettant 2 cercles focaux donnés, d'espèce différentes, ont une même excentricité, bien déterminée, dont l'auteur donne diverses expressions. Les 2 coniques de la famille passant par un point donné M sont déterminées. La généralisation au cas, projectivement équivalent, des coniques bitangentes à 2 coniques fixes, est également étudiée. Les coniques admettant 2 cercles focaux donnés, de même espèce, ont leurs foyers, non situés sur l'axe de symétrie commun, qui décrivent le cercle de similitude des deux cercles focaux donnés. L'auteur détermine les 2 coniques de la famille qui passent par un point donné, et la conique de la famille qui admet un centre donné.

P. Belgodère (Paris).

Noble, C. E. An anallagmatic cubic. Amer. Math. Monthly 55, 7–14 (1948).

Sen Gupta, B. K. On the Cayleyan of a non-singular cubic. Math. Student 14, 20–22 (1946).

Taylor, D. G. On certain configurations of congruent triangles. Math. Gaz. 31, 270–278 (1947).

Given a triangle ABC, the author shows that it is possible to construct ten triangles inscribed in ABC and ten triangles circumscribed about ABC, all congruent to ABC. Properties of both the inscribed and the circumscribed triangles are pointed out. Some of these triangles are related to the remarkable points of the basic triangle, like the circum-

center, the orthocenter, the Brocard points. The author gives several helpful figures. *N. A. Court.*

Goormaghtigh, R. On the two-angle pole of a line to a triangle. Amer. Math. Monthly **55**, 71–75 (1948).

Goormaghtigh, R. Sur certaines ellipses associées au triangle. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. **17**, 123–126 (1947).

Falch, O. Apollonius' problem. Norsk Mat. Tidsskr. **29**, 104–105 (1947). (Norwegian)

Convex Domains, Extremal Problems

Hadwiger, H. Nonseparable convex systems. Amer. Math. Monthly **54**, 583–585 (1947).

As an extension of a result of A. W. Goodman and R. E. Goodman [Amer. Math. Monthly **52**, 494–498 (1945); these Rev. **7**, 164] the author proves the following proposition. Consider in the plane a system of n convex domains K_i $(i=1, \cdots, n)$ such that no two of them can be separated by a straight line which intersects none of the domains K_i. Let K_0 be the convex hull of the system and denote the circumference, the diameter and the radius of the circumcircle of K_i $(i=0, \cdots, n)$ by C_i, D_i and R_i, respectively. Then $C_0 \leqq \sum_{i=1}^{n} C_i$, $D_0 \leqq \sum_{i=1}^{n} D_i$, $R_0 \leqq \sum_{i=1}^{n} R_i$. Concerning the circumferences an analogous inequality is also proved for spherical geometry. *L. Fejes Tóth* (Budapest).

Matumura, Sôzi. Über Flächen und Kurven. XXVI–XXXVI. Einige Bemerkungen über Eilinien und Eiflächen. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. **29**, 159–173, 309–342 (1941), 441–508 (1942); **30**, 47–56 (1942), 87–101, 117–130, 143–152, 163–174, 187–195, 213–227, 241–253 (1943).

Matumura, Sôzi. Über Flächen und Kurven. XXXVII–XLV. Einige Bemerkungen über Eilinien und Eiflächen. Mem. Fac. Sci. Taihoku Imp. Univ. Ser. I. **1**, 7–12 (1943), 17–22, 29–36, 45–50, 57–59, 67–68, 77–78, 83–84, 91 (1944).

Continuation of a series of miscellaneous notes. For previous installments cf. these Rev. **4**, 111.

Busemann, Herbert. The isoperimetric problem in the Minkowski plane. Amer. J. Math. **69**, 863–871 (1947).

Let $F(x, y)$ be a continuous function, positive for $(x, y) \neq (0, 0)$ and positively homogeneous of order 1, C the indicatrix $F(x, y)=1$. The isoperimetric problem in question is the following: to find among all simple (oriented) closed curves $D: x(t), y(t)$ with a given F-length one which bounds the greatest (Euclidean) area $A(D)$. The F-length of D is interpreted as the mixed area $A(D, K)$ of D and the polar reciprocal K of C with respect to the unit circle rotated through $\pm\pi/2$. It follows from the Brünn-Minkowski theory that $A(D)$ is maximum when D is homothetic to K. This property holds if C is not convex if we replace C by its convex closure $\bar{C}: \bar{F}(x, y)=1$. A necessary and sufficient condition for the Minkowski circles to solve the isoperimetric problem is the symmetry of the orthogonality relation. The relation between the F-length λ_F and the $\bar{F}$-length $\lambda_{\bar{F}}$ is $\lambda_{\bar{F}}(G)=$ lower limit of λ_F in G. [Reviewer's note. The results implying the replacement of the indicatrix by its convex closure can be easily derived from L. C. Young's theory. See Proc. Roy. Soc. London. Ser. A. **141**, 325–341 (1933); C. R. Soc. Sci. Lett. Varsovie. Cl. III. **30**, 211–234 (1937).] *C. Y. Pauc* (Cape Town).

***Busemann, Herbert. On the problem of Dido.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 63–73. Interscience Publishers, Inc., New York, 1948. $5.50.

Let $F(x, y; \dot{x}, \dot{y})$ be continuous in the four variables, positive for $(\dot{x}, \dot{y}) \neq (0, 0)$, positively homogeneous of degree 1 in $(\dot{x}, \dot{y})$ and symmetric in $(\dot{x}, \dot{y})$, that is

$$F(x, y; \dot{x}, \dot{y}) = F(x, y, -\dot{x}, -\dot{y}),$$

and let H be the indicatrix $F(x, y)=1$, $m(x, y)$ a positive function which is summable over any bounded domain; $M(C)$ denotes the integral $\int m(x, y)dxdy$ extended over the closed domain $B(C)$ bounded by a simple closed curve. The problem of Dido for F, m and a fixed number L_0 is: find among all simple closed curves C with F-length $=L_0$ those which maximize $M(C)$. The author makes systematic use of the convexity principle, stated as follows: if K is a solution of the problem of Dido for F, m and L_0, then $B(K)$ is geodetically convex. From it he derives in the case of a Minkowskian integrand $F(\dot{x}, \dot{y})$ a deeper principle, the regularity principle: if the indicatrix $H^*: F^*(x, y)=1$ is not convex and $F(x, y)$ is the integrand whose indicatrix is the boundary H of the convex closure of H^*, then any solution K of the problem of Dido for F, m and L_0 is also a solution for F^*, m and the same L_0. The regularity principle permits the restriction to quasi-regular integrands. From the convexity principle applied to an integrand $F(\dot{x}, \dot{y})$ follows immediately: if H is a (convex) polygon with vertices $p_1, \cdots, p_n$, then a solution of the problem of Dido for any m and L_0 is a convex polygon whose sides are parallel to the radii p_iO of H. A relation (R) between consecutive angles and sides of such a convex polygon is deduced in an elementary way from the maximum condition; the proof requires the continuity of m. If $\rho(\theta)$ is a function with continuous second derivatives, the indicatrix $H: r=\rho^{-1}(\theta)$ is approximated by the polygon H_n whose vertices have polar coordinates $\rho_i^{-1}=\rho^{-1}(2\pi i/n)$, $2\pi i/n$, $i=1, 2, \cdots, n$, n even. A passage to the limit $(n \to \infty)$ applied to H_n and (R) yields the following theorem. Let solutions K and K^* of the problem of Dido for F and m and of the isoperimetric problem (i.e., $m=1$) for F have radii of curvature $R(\theta)$ and $R^*(\theta)$ at the points where the (oriented) tangents of K and K^* respectively have direction θ. If $m(\theta)$ denotes the value of $m(x, y)$ at the same point of K, then $m(\theta)R(\theta)=K \cdot R^*(\theta)$, K denoting a constant. The relation (R) applied to the isoperimetric problem $(m=1)$ yields a new proof of the Brunn-Minkowski theorem that in the paper reviewed above was the main tool for solving this same problem. The paper concludes with the indication of an example for the regularity principle in n dimensions. The considerations throughout this paper belong to elementary geometry and calculus. Rigor is missed at places as the author himself observes. It seems that the difficulty of setting up simple and rigorous proofs is a general feature of the direct geometrical methods at their present stage. *C. Y. Pauc* (Cape Town).

Pleijel, Arne. A proof of the isoperimetric property of the circle. Mat. Tidsskr. B. **1947**, 35–36 (1947). (Danish)

A new proof based on Crofton's integral which expresses the arc bounded by a convex curve as an average over the length of its chords. *H. Busemann.*

Davis, Chandler. The short-cut problem. Amer. Math. Monthly **55**, 147–150 (1948).

The author investigates the problem of finding the shortest path between two points in a network of straight lines.

P. Erdös (Syracuse, N. Y.).

Algebraic Geometry

Motzkin, Th., and Robinson, A. The characterization of algebraic plane curves. Duke Math. J. **14**, 837–853 (1947).

Let all coordinates lie in a commutative field F of arbitrary characteristic and let l_∞ be the line at infinity in the projective plane. Defining distances suitably with respect to l_∞, and the cyclic ratio of a point-set S for a triangle T as the product of the ratio in which the points of S on the sides of T divide the sides, the authors prove that algebraic curves are completely characterized as sets (not the null set or the whole plane) for which the multiplicity of every point of the set on every line through it can be defined to satisfy: (A) the set has a constant finite number of intersections with every line not contained in it; (B) the cyclic ratio of the set is 1 for every triangle with no vertex on the set or at infinity. Plane sets which are not algebraic curves are constructed to satisfy (A) or (B) alone. In particular, a point-set is seen to exist which "behaves even better" than any algebraic curve, having n distinct points in common with every line. *D. Pedoe* (London).

Abellanas, Pedro. On the geometrical theory of algebraic surfaces for a perfect coefficient field of characteristic p. Revista Acad. Ci. Madrid **36**, 482–499 (1942). (Spanish)

Some results of Zariski's in the theory of algebraic surfaces over ground fields of characteristic zero are extended to perfect ground fields of characteristic p. In particular, it is shown that the geometric definition of a simple point of a surface (in terms of the multiplicity of the intersection with two generic primes) is equivalent to the arithmetic definition [O. Zariski, Amer. J. Math. **61**, 249–294 (1939), p. 251]. It is further pointed out that, with Zariski's definition of normality, it is still true that the singular locus of a normal surface is of dimension greater than or equal to 0 [loc. cit., p. 285]. *D. B. Scott* (London).

Brusotti, Luigi. Premesse topologiche allo studio dei fasci reali di curve algebriche sopra una superficie algebrica reale. Ann. Mat. Pura Appl. (4) **25**, 67–109 (1946).

This paper is concerned with the fundamental topological considerations which underlie the theory of real pencils of curves on a real algebraic surface. The basic notion is that of a pencil of circuits on a surface, that is, a system of circuits on a surface which has the property that through each point of the surface (with at most a finite number of exceptions) there passes just one circuit of the system, the circuits being represented by the points of a graph in the following way. The points of an open 1-cell represent (continuously) circuits without singular points, and the vertices represent "extremal" circuits in which singular points may appear. Even in the simplest case there are several types of extremes to be considered. The permissible graphs have no vertices common to more than three 1-cells, and according to the number of 1-cells through a vertex there are restrictions on the possible type of extremal circuit represented by it. The graph can be "annotated" to give a complete description (to within homeomorphism) of the pencil of circuits. Certain results on the topology of the surface are deduced from the annotated graph of a pencil of circuits on it. The ideas are extended to pencils of "graphical" curves (essentially finite sets of circuits) and thence to pencils of real curves on a surface. *W. V. D. Hodge.*

Enriques, Federigo. Sur le théorème de Riemann-Roch concernant les surfaces algébriques et sur les systèmes des courbes canoniques et pluricanoniques. Revista Acad. Ci. Madrid **40**, 149–159 (1946).

These results, discovered in 1941, fall under several headings. All general theorems deal with regular algebraic surfaces for which $p_g=p_a=p>0$. (1) Fixed part of the canonical system. It was thought that by a suitable birational transformation the fixed part of the canonical system on an algebraic surface can always be transformed into a simple point; that is, the fixed part is always an exceptional curve with virtual genus 0, virtual grade -1 [M. Noether, Math. Ann. **8**, 495–533 (1875)]. A counterexample suggested by Castelnuovo to Enriques [Mem. Soc. Ital. Sci. (3) **10**, 1–81 (1896), p. 72] of a quintic surface with three tacnodes whose plane meets the surface in a cubic curve and a conic touching at 3 points was generally accepted until the reviewer showed [Proc. Cambridge Philos. Soc. **31**, 536–542 (1935)] that the conic is an exceptional curve, and the cubic the canonical curve of the surface. Effective counterexamples were then described [B. Segre, Rend. Sess. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. **40**, 43–46 (1936); L. Campedelli, Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **71**, 370–390 (1936); J. Bronowski, J. London Math. Soc. **12**, 140–145 (1937); Proc. Cambridge Philos. Soc. **39**, 31–34 (1943); these Rev. **4**, 168]. Enriques gives several counterexamples. Two appear in Bronowski's second paper.

(2) The Riemann-Roch theorem. It is known that every nonspecial linear system of irreducible curves C on a surface has dimension $r\geqq p+n-\pi+1$, n and π denoting the order and genus of C. Systems for which the equality sign holds are called regular. Enriques proves that the system adjoint to a connected curve L, composed of simple or multiple parts, is always regular.

(3) The irreducibility of the variable parts of curves of the bicanonical system. In order to prove that the canonical system always consists of connected curves when $p>0$, $p^{(1)}>1$ ($p^{(1)}$ is the linear genus of the surface), Enriques proves that even if the bicanonical system is reducible, the variable parts of the bicanonical curves are irreducible.

(4) Regularity of the bicanonical system. It follows that for a surface with $p>0$, $p^{(1)}>1$, the adjoint of the canonical system, which is the bicanonical system, is regular. Hence all the pluricanonical systems are regular.

(5) Simplicity of the pluricanonical systems. It is stated that, with two exceptions, the double plane with branch curve of order 8, and the double plane with branch curve of order 10 with two infinitely near quintuple points, all regular surfaces with $p>0$, $p^{(1)}>1$ and order not exceeding $q(p^{(1)}-1)$ have simple tricanonical systems. *D. Pedoe.*

Pompilj, Giuseppe. Su alcune equivalenze funzionali. Ann. Mat. Pura Appl. (4) **25**, 135–153 (1946).

L'auteur généralise une étude de Severi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **17**, 876–881 (1933)] sur l'équivalence fonctionnelle d'une courbe unie dans une

correspondance à valence sur une surface. Soit T une correspondance algébrique de S_r en lui-même, la variété des couples de points associés ayant la dimension $r+h$. Lorsque la variété des points unis a une dimension supérieure à h (incidences surabondantes) il y a lieu de déterminer l'équivalence fonctionnelle de toute composante D_{h+i} de dimension $h+i$, pour la variété virtuelle de dimension h lieu des points unis.

On appelle droite principale toute droite d'accumulation dans l'ensemble des droites qui joignent deux points distincts de S_r associés dans T: le lieu des droites principales issues d'un point de D_{h+i} est une variété V_{r-i}. Sous réserve de compter ensuite D_{h+i} avec une certaine multiplicité, l'auteur se limite au cas où V_{r-i} est un espace linéaire S_{r-i}. Soit M_h^j la variété lieu des points de D_{h+i} qui appartiennent à un S_{r-j} générique et possèdent une droite principale incidente à un S_j générique de S_{r-j}. L'auteur démontre en examinant d'abord le cas de la surabondance $s=1$ puis par récurrence que l'équivalence fonctionnelle Z_h de D_{h+i} est définie par la relation: $Z_h=M_h^0+M_h^1+\cdots+M_h^i$.

Dans une seconde partie l'auteur applique ce résultat à la détermination de l'équivalence fonctionnelle d'une variété unie D_{h+i} pour une correspondance sur une variété V_r, puis à la détermination de l'équivalence fonctionnelle d'une variété D_{h+i} comme composante (de dimension surabondante) de l'intersection de deux variétés V_h et V_{r-h+k} de S_r. Dans ce dernier cas l'espace principal S_{r-i} est l'espace tangent commun à V_h et V_{r-h+k}. Il serait intéressant de savoir donner aux résultats, strictement projectifs, de cette note, un aspect invariantif intrinsèque. *L. Gauthier.*

*Lunell, Einar. **Liniengeometrische Studien, mit besonderer Rücksicht auf Regelflächen mit vollständigem Zerfalle der Doppelkurve.** Dissertation, University of Uppsala, 1940. 162 pp.

Der Verfasser ist ein Schüler von A. Wiman und die Dissertation ist angeregt durch dessen liniengeometrische Arbeiten. Kap. I enthält eine ausführliche Übersicht über die verschiedenen Abbildungsmethoden in der Liniengeometrie, wobei mit gutem Erfolg versucht wird diese möglichst einheitlich zusammen zu fassen. Mit Hilfe des Kleinschen Bildes in S_5 kann jede Abbildung als eine birationale Punkttransformation betrachtet werden und in dieser Weise erhält man liniengeometrische Deutungen der älteren und neueren Rationalitätskriterien für Punktmannigfaltigkeiten. Wir nennen noch eine Aussage des Verfassers [S. 14] nach welcher ein Komplex n-ter Ordnung mit ∞^3 Doppellinien für $n>2$ nicht notwendig ein spezieller ist. (Beispiel für $n=3$: $Ap_{12}^2+Bp_{12}p_{34}+Cp_{34}^2$, wo A, B und C lineare Formen in den Linienkoordinaten p_{ij} sind.) Von Bedeutung ist auch ein auf Fano und Enriques zurückgehender Satz: eine M_3, deren Koordinaten gleich Polynomen dreier Parameter gesetzt worden sind, ist nicht notwendig rational.

Der Verfasser definiert die Hauptkorrespondenz T_h einer Regelfläche als diejenige, in welcher zwei Erzeugende, die sich schneiden, einander zugeordnet sind. Für eine Fläche n-ter Ordnung ist T_h eine involutorische $(n-2, n-2)$-Korrespondenz mit der Wertigkeit 2. Die projektive Theorie der algebraischen Regelflächen wird vom Verfasser im wesentlichen als ein Problem dieser Hauptkorrespondenz betrachtet. Kap. II gibt, gestützt auf die Castelnuovoschen Ergebnisse für das Maximalgeschlecht einer Kurve, Sätze über das grösstmögliche Geschlecht der Regelflächen.

In der Wimansche Dissertation [Lund, 1892] über die Regelflächen siebenter Ordnung wird die Doppelkurve auf eine Bisekantenregelfläche abgebildet. In Kap. III betrachtet der Verfasser anschliessend allgemein Punktkorrespondenzen auf Raumkurven und die zugehörigen Bisekantenregelflächen unter Berücksichtigung der bezüglichen Literatur. In Kap. IV wird der Zerfall der Hauptkorrespondenz in dem extremen Fall untersucht, wo dieser Zerfall vollständig ist. Die Doppelkurve besteht dann aus $n-2$ Leitkurven. Dieses Kapitel und das letzte über vorkommende Typen von Regelflächen mit vollständigem Zerfalle der Hauptkorrespondenz bilden den wichtigsten Teil der Arbeit und sie erhalten eine Fülle von Eigenschaften der Regelflächen höherer Ordnungen. Wir weisen zum Beispiel hin auf eine Regelfläche der Ordnung acht vom Geschlecht eins, ohne Leitgerade, deren Doppelkurve und Doppeldeveloppabele beide in sechs getrennte Teile zerfallen. *O. Bottema* (Delft).

Lesieur, L. Un problème de géométrie énumérative de l'espace à huit dimensions. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 542–547 (1947).

Bekanntlich enthält die Fläche F_2 des R_4, in der sich 2 allgemeine Quadriken schneiden, 16 Gerade (F_2 wird in eine Fläche 4. Grades der R_3 mit Doppelkegelschnitt projiziert). Verfasser untersucht den hieran sich anschliessenden Fall einer Schnittmannigfaltigkeit allgemeiner Quadriken, auf der endlich viele Ebenen liegen und zählt die Zahl derselben ab. Dieser Fall tritt ein bei der Schnitt-M_6 dreier Quadriken des R_8. Die Anzahl 1024 der auf einer solchen M_6 gelegenen Ebenen wird dadurch erhalten, dass man für die Gesamtheit der Ebenen des R_8 gewisse einfache durch Schubertsche Symbole zu bezeichnende Basisgesamtheiten angibt und ausrechnet, wie sich die Menge aller Ebenen einer Quadrik in dieser Basis ausdrückt, wobei nur ein Gleid dieser Basis einen von Null verschiedenen Beitrag liefert. Die Zahl der 3 Quadriken gemeinsamen Ebenen wird dann im wesentlichen auf die gleiche Aufgabe für die Basis zurückgeführt. *W. Burau* (Hamburg).

Togliatti, Eugenio. Alcune osservazioni sulle superficie razionali che rappresentano equazioni di Laplace. Ann. Mat. Pura Appl. (4) **25**, 325–339 (1946).

Ce mémoire fait suite à un autre du même auteur sur le même sujet [Comment. Math. Helv. **1**, 255–272 (1929)]. Une première partie est consacrée à quelques remarques d'ordre général. L'auteur démontre notamment que la surface lieu des points communs à deux S_n osculateurs d'une C^{2n} rationnelle normale de S_{2n}, surface d'ordre n^2, représente l'équation de Laplace: $(\alpha-\beta)\theta_{\alpha\beta}+n\theta_\alpha-n\theta_\beta=0$. Les caractéristiques sont les osculantes d'ordre n de C^{2n}, qui forment sur F un système continu unique d'indice 2; F est représentée sur le plan par le système de toutes les C^n apolaires à une enveloppe de seconde classe non dégénérée dont les tangentes représentent les caractéristiques.

La seconde partie est consacrée à la détermination de toutes les surfaces représentables sur le plan au moyen de systèmes linéaires de cubiques, et qui représentent une équation de Laplace. Outre la solution obtenue par l'énoncé précédent pour $n=3$: F^6 de S_6, et ses projections sur S_5, la réglée R^5 de S^5 représentée par les C^3 ayant un point double donné, il y a une surface F^9 de S_9, une surface F^5 de S_5 et une surface F^4 de S_4. *L. Gauthier* (Nancy).

Derwidué, L. Remarque sur les variétés algébriques à surfaces-sections rationnelles. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 635–637 (1947).

Godeaux, Lucien. Surfaces dont le système canonique appartient à une involution. Bull. Soc. Roy. Sci. Liége 16, 7–9 (1947).

Godeaux, Lucien. Sur quelques surfaces-enveloppes du huitième ordre. Bull. Soc. Roy. Sci. Liége 16, 9–17 (1947).

Godeaux, Lucien. Les surfaces algébriques irrégulières. Revue Sci. 85, 812–816 (1947).

Godeaux, Lucien. Sur la courbe canonique de genre six. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 33, 611–616 (1947).

Lorent, H. Contribution à l'analogie entre les cubiques planes de genre un et les biquadratiques gauches de première espèce. VI. Bull. Soc. Roy. Sci. Liége 16, 78–83 (1947).

For preceding parts cf. the same Bull. 12, 136–140, 204–207, 508–513, 632–636, 674–680 (1943); these Rev. 7, 72.

Metelka, Josef. Sur certains systèmes linéaires surabondants de courbes planes. Bull. Soc. Roy. Sci. Liége 16, 94–97 (1947).

Seifert, L. The hypersurface of the third degree with a bispatial point in four-dimensional space. Rozpravy II. Třídy České Akad. 55, no. 7, 19 pp. (1945). (Czech)

Stubban, John Olav. Sur l'involution de Bertini. Bull. Soc. Roy. Sci. Liége 16, 98–102 (1947).

Differential Geometry

Kasner, Edward, and De Cicco, John. Physical families of curves in space. Proc. Nat. Acad. Sci. U. S. A. 34, 68–72 (1948).

In the terminology of the authors, a system of curves S_k, in three-dimensional space and in an arbitrary field of force, consists of the curves along which constrained motion is possible such that the osculating plane at each point contains the force vector, and such that the pressure P is proportional to the normal component N of the force ($P=kN$). For k equal to 0, -2, 1 or ∞, S_k is the system of trajectories, the system of brachistochrones, the system of general catenaries or the system of velocity curves, respectively. In this note the authors state several properties of the systems of curves S_k. Among these properties are the following. (1) The osculating planes of the ∞^3 curves passing through a given point form a pencil. (2) The osculating spheres of the ∞^1 curves passing through a given point in a given direction form a pencil. (3) Consider the curves which pass through a given point in the direction of the line of force through that point. All of these have zero curvature at the given point, except one particular curve. The ratio ρ of the curvature of this curve to the curvature of the line of force at the point is $\rho=(k+1)/(k+3)$.

L. A. MacColl (New York, N. Y.).

Vaona, Guido. Sui flessi di specie superiore delle curve piane. Boll. Un. Mat. Ital. (3) 2, 117–123 (1947).

Differential elements of the type $y=a_0x^k+\cdots$ ($k\geqq 4$, $a_0\neq 0$) have been studied by E. Bompiani [same Boll. (2) 5, 156–168 (1943); these Rev. 7, 392]. The author gives a new construction of their invariants using the polars of the origin with respect to algebraic curves containing those elements, and application to the Hessian of a quartic curve.

E. Bompiani (Rome).

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. XLII–LII. Mem. Fac. Sci. Agric. Taihoku Imp. Univ. 29, 97–158, 175–308 (1941), 349–439 (1942); 30, 1–46 (1942), 57–85, 103–116, 131–142, 153–161, 175–186, 197–212, 229–239 (1943).

Matumura, Sôzi. Beiträge zur Geometrie der Kreise und Kugeln. LIII–LXI. Mem. Fac. Sci. Taihoku Imp. Univ. Ser. I. 1, 1–6 (1943), 13–16, 23–28, 37–43, 51–56, 61–66, 69–75, 79–82, 85–89 (1944).

Continuation of a series of miscellaneous notes. For previous installments cf. these Rev. 4, 53.

Lalan, Victor. La représentation conforme minima singulière et la transformation d'Hazzidakis. C. R. Acad. Sci. Paris 226, 383–384 (1948).

A conformal mapping of two surfaces is called minimal if it preserves the pseudo-arcs of the minimal lines; it is called singular if moreover it preserves the pseudo-geodesic curvatures of these lines. In this note the author develops means of finding two surfaces S and $\bar{S}$ which are related in this fashion. This is accomplished by starting from a pair of surfaces Σ and $\bar{\Sigma}$ related according to a formula of Hazzidakis. By definition Σ and $\bar{\Sigma}$ have constant total curvature m^2n^2. The surface S is taken to be that surface which is parallel to Σ at a distance $1/n$; $\bar{S}$ is then the parallel to $\bar{\Sigma}$ at a distance $-1/m^2n$. From known relationships between Σ and $\bar{\Sigma}$ the author obtains various connections between S and $\bar{S}$.

C. B. Allendoerfer (Haverford, Pa.).

Lalan, V. Les surfaces envisagées dans leurs rapports avec leurs lignes minima. Bull. Soc. Math. France 75, 63–88 (1947).

In the past several years the author has devoted himself to the study of those properties of ordinary surfaces which are most closely connected with the minimal lines of the surface. Before the present paper these results appeared as brief statements of isolated theorems [C. R. Acad. Sci. Paris 222, 632–633 (1946); 223, 569–570, 707–709, 883–885 (1946); 224, 518–520, 1201–1203 (1947); these Rev. 7, 393; 8, 228, 350, 404, 530; cf. also the following review]. This paper, however, is the first of several connected expositions of Lalan's research which are due to appear, and is the paper which should be read at the outset by those wishing to understand this theory.

The content of this paper is restricted to the development of the analytic machinery needed for the theory to come. Writing in the Cartan idiom, the author begins by defining an appropriate moving trihedral, the "bi-isotropic trihedral," of the surface. The first two vectors I_1 and I_2 of this trihedral at a point are tangent vectors taken in the minimal directions at this point; the third vector I_3 is the unit normal vector. Excluding umbilical points from the discussion, the author normalizes the lengths of I_1 and I_2 in a convenient fashion, and finally writes the two fundamental forms of the surface as follows:

$$ds^2=2\omega_1\omega_2/A;\quad \Phi=\omega_1^2+2H\omega_1\omega_2/A+\omega_2^2,$$

where H is the mean curvature and A is the asphericity;

i.e., one-half the difference of the principal curvatures. For a related treatment of these quantities not mentioned by Lalan see H. W. Alexander [Trans. Amer. Math. Soc. **47**, 230–253 (1940); these Rev. **1**, 269] and T. Y. Thomas [Bull. Amer. Math. Soc. **51**, 390–399 (1945); these Rev. **7**, 30].

The remainder of the paper is devoted to the study of the properties of this trihedral. Among the topics covered are the formula for the differential of the pseudo-arc of the minimal curves, a formulation of the Gauss and Codazzi equations, relations between this trihedral and minimal coordinates, relations between this trihedral and that of Darboux, and expressions for the differential parameters of Beltrami. Finally the author turns to a subject of major importance in the later part of his theory, namely the properties of surfaces with isothermal mean curvature. By definition these surfaces are such that the curves along which the mean curvature is constant are isothermal lines. Only a brief treatment of these surfaces is given here, the full theory being deferred to later papers.

C. B. Allendoerfer (Haverford, Pa.).

Lalan, Victor. Les surfaces d'Ossian Bonnet en tant que surfaces à courbure moyenne isotherme. C. R. Acad. Sci. Paris **226**, 214–216 (1948).

In the spirit of his earlier work [see the preceding review] Lalan observes that the surfaces of Bonnet are special cases of Lalan's surfaces of isothermal mean curvature, and he characterizes them analytically in terms of the formalism described in the preceding paper. No proofs are given.

C. B. Allendoerfer (Haverford, Pa.).

Maneng, Louis. Sur l'équation de Pfaff équivalente aux deux équations de Codazzi. C. R. Acad. Sci. Paris **226**, 550–552 (1948).

Lalan [same C. R. **224**, 518–520 (1947); these Rev. **8**, 404] obtained a formulation of the equations of Codazzi as a single total differential equation by the use of the biisotropic trihedral [see the second preceding review]. The present paper arrives at an equivalent result using differential forms related to those appearing in the theory of the trihedral of Darboux and without introducing imaginary quantities such as appear in Lalan's work.

C. B. Allendoerfer (Haverford, Pa.).

Wu, Ta-Jen. On pairs of curves in non-Euclidean space. Acad. Sinica Science Record **2**, 31–36 (1947).

Notations: E_3: three-dimensional elliptic space with measure of curvature $1/k^2$, $k>0$; H_3: three-dimensional hyperbolic space with measure of curvature $-1/k'^2$, $k'>0$. For a curve C of E_3 or H_3, the tangent, the principal normal, the binormal are denoted by t, p, b, respectively, their polar lines by t^*, p^*, b^*. If $\bar{C}$ is a second curve of E_3 or H_3, the corresponding lines for $\bar{C}$ are denoted by the same symbols with bars. The author investigates pairs of curves such that one of the lines t, p, b, t^*, p^*, b^* coincides with one of the lines $\bar{t}$, $\bar{p}$, $\bar{b}$, $\bar{t}^*$, $\bar{p}^*$, $\bar{b}^*$. He gives necessary and sufficient conditions for a curve C to possess an associated curve in that sense and results concerning the differential properties of the curves C and $\bar{C}$ in two corresponding points. The following nontrivial cases are treated: $t=\bar{t}^*$ (polar curves), $p=\bar{p}$ (Bertrand curves), $b=\bar{b}$, $p=\bar{b}$, $b=\bar{p}$, $p=\bar{b}^*$. No proofs are given.

C. Y. Pauc (Cape Town).

Charrueau, André. Sur la déformation infiniment petite et sur des congruences qui s'y rattachent. Bull. Soc. Math. France **74**, 42–58 (1946).

Soient x, y, z les coordonnées orthogonales de l'espace et (1) $z=f(x,y)$ une surface non développable. La solution (2) $z_1=f_1(x,y)$ de l'équation

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f_1}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial^2 f_1}{\partial x\partial y}=0 \tag{3}$$

effectue le problème de la déformation infiniment petite de la surface (1). Soit Q le point $(0,0,-1)$ du demi-axe négatif des z et QM la droite parallèle à la normale n construite au point $P(x,y,z)$ de (1) et QM_1 la droite parallèle à la normale n_1 de (2) construite au point $P_1(x,y,z_1)$. Les points M, M_1 sont des points du plan $z=0$. Menons par M la droite d_1 parallèle à n_1 et par M_1 la droite d parallèle à n. Quand le point $(x,y,0)$ se déplace dans le plan $z=0$, les droites d et d_1 engendrent respectivement les congruences C et C_1. Chacune de ces congruences a pour surface moyenne le plan $z=0$. L'auteur étudie les propriétés différentielles des surfaces (1) et (2) et aussi des congruences C et C_1.

Pour les surfaces (1) et (2) avec orthogonalité des éléments linéaires l'auteur modifie le problème et étudie le lieu S des points (x_1,y_1,z) et le lieu S_1 des points (x,y,z_1) pour lesquels $x_1=x_1(x,y)$, $y_1=y_1(x,y)$, $z_1=z_1(x,y)$ et

$$dx_1=-\frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x}dx+\left(C_1-\frac{\partial f}{\partial y}\frac{\partial f_1}{\partial x}\right)dy,$$

$$dy_1=-\left(C_1+\frac{\partial f}{\partial x}\frac{\partial f_1}{\partial y}\right)dx-\frac{\partial f}{\partial y}\frac{\partial f_1}{\partial y}dy,$$

avec

$$\frac{\partial C_1}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial f_1}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial f_1}{\partial x}\right),$$

$$\frac{\partial C_1}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial f_1}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial f_1}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f_1}{\partial x^2}-2\frac{\partial^2 f}{\partial x\partial y}\frac{\partial^2 f_1}{\partial x\partial y}=0.$$

En partant de S et S_1 l'auteur construit les deux congruences $\bar{C}$ et $\bar{C}_1$ de manière analogue à C et C_1. Aussi $\bar{C}$ et $\bar{C}_1$ admettent le plan $z=0$ comme surface moyenne. L'auteur démontre que chacune de ces congruences peut être définie au moyenne d'une seule surface spéciale et étudie la correspondance entre les asymptotiques de cette surface et des développables de la congruence prise en considération.

F. Vyčichlo (Prague).

Ryžkov, V. V. On a question about the projective deformation of congruences. Doklady Akad. Nauk SSSR (N.S.) **59**, 17–20 (1948). (Russian)

L'auteur rapporte la congruence étudiée à son tétraèdre de Wilczynski et cherche si la déformation projective peut ou non dépendre de fonctions arbitraires; il obtient ainsi les deux résultats suivants. Premièrement, pour une congruence qui n'est pas W et dont les nappes focales ne sont dégénérées ni ponctuellement ni tangentiellement, la déformation ne peut dépendre que de constantes arbitraires. Deuxièmement, les seules congruences à surfaces focales non dégénérées, dont la déformation dépend de fonctions arbitraires, sont les congruences W à surfaces focales réglées et les congruences projectivement applicables sur elles.

Si l'on passe à une congruence W quelconque, cette congruence est projectivement applicable sur sa congruence transformée par dualité, mais peut-être pas par déformation projective continue; pourtant, toute congruence C projectivement applicable sur une congruence C_1 appartenant à un complexe linéaire est déformable d'une façon continue en cette congruence C_1 ou en la transformée C_2 de C_1 par dualité; ce résultat est valable pour toute congruence Γ applicable sur une congruence C à surfaces focales réglées; Γ possède ∞^2 complexes linéaires osculateurs, C en possède ∞^1; Γ s'applique aussi sur une congruence C_1 appartenant à un complexe linéaire, laquelle n'a qu'un complexe linéaire osculateur. Par suite, au point de vue de la déformation projective, les congruences à nombre égal de complexes linéaires osculateurs ne forment pas une classe fermée, comme dans le cas particulier d'É. Cartan. *B. Gambier.*

Backes, Fernand. Congruences W déduites d'un réseau de lignes de courbure. C. R. Acad. Sci. Paris 226, 385–386 (1948).

Homogeneous coordinates (X, Y, Z, T) of a point $P(x, y, z)$ are defined by

$$X=-x+y+z+h, \quad Z=x+y-z+h,$$
$$Y=x-y+z-h, \quad T=-x-y-z+h$$

(h=constant) and Plücker line coordinates of a line l by

$$p_{12}=X^2+Y^2+Z^2+T^2, \quad p_{13}=X+iY, \quad p_{14}=Z+iT,$$
$$p_{34}=-1, \quad p_{42}=X-iY, \quad p_{23}=Z-iT.$$

If P generates a surface S referred to its lines of curvature, then $1, x, y, z, x^2+y^2+z^2$, and p_{ij} are solutions of a differential equation of the form $\theta_{uv}=A\theta_u+B\theta_v$; hence l generates a W congruence whose developables correspond to the minimal net on S. Any transformation of S preserving the lines of curvature induces a transformation of the W congruence into a W congruence. *V. G. Grove* (East Lansing, Mich.).

Anglade, E. Sur les surfaces dont la suite de Laplace adjointe se termine suivant les cas de Laplace et de Goursat. Bull. Soc. Math. France 75, 43–48 (1947).

Let u=constant, v=constant be the asymptotic curves on a surface S in a projective space S_3. To the point P on S may be made to correspond two points U, V in a projective space S_5, the coordinates of U and V being the Plücker line coordinates of the tangents to v=constant, u=constant, respectively. To the surface S may be associated a sequence of Laplace

$$\Sigma(\cdots, U_2, U_1, U, V, V_1, V_2, \cdots)$$

in S_5. The present note gives some results in the case in which Σ terminates at U_n in the sense of Laplace. A few of these may be described as follows. On the surface U_{n-1} the osculating planes to u=constant as U_{n-1} varies along v=constant all pass through a fixed space of three dimensions whose image in S_3 is a linear complex with directrices g, g'. So to each asymptotic curve v=constant are associated two lines g, g'. As v varies, the lines g, g' generate the focal surfaces of a congruence W_g of Weingarten. The congruences osculating these focal surfaces along g and g' both belong to the complex U_n. There is associated with U_n a linear congruence with directrices f and f'. These lines are tangent to the surfaces generated by g and g', and as v varies, f and f' generate the focal surfaces of a second congruence W_f of Weingarten. The lines of these ruled focal surfaces belong to W_g. *V. G. Grove.*

Kovancov, N. I. On surfaces on which a line of the canonical bundle coincides at every point with the metric normal. Doklady Akad. Nauk SSSR (N.S.) 58, 1261–1263 (1947). (Russian)

Il y a encore peu de travaux où certaines notions métriques sont rattachées à des notions projectives; ici l'auteur traite la question: trouver les surfaces dont la normale métrique en chaque point est une droite du faisceau canonique. La surface S est rapportée à ses asymptotiques u, v; les axes Ox, Oy sont les tangentes asymptotiques au point M_0 et Oz est la normale métrique. Les deux formes quadratiques fondamentales sont

$$ds^2=A^2du^2+2AC\cos w\,dudv+C^2dv^2, \quad 2D'dudv.$$

La première directrice de Wilczynski a pour paramètres directeurs

$$l=\frac{A}{2D'}\left(2b-a'-\frac{\partial\log c}{\partial v}\right), \quad m=\frac{C}{2D'}\left(2b'-a-\frac{\partial\log c'}{\partial u}\right), \quad n=1,$$

où $a=G_{11}^1$, $b=G_{12}^1$, $c=G_{11}^2$, etc. sont les parenthèses de Christoffel. L'équation de S au voisinage de M_0 a la forme

$$z=a_{11}xy+\tfrac{1}{6}(a_{30}x^3+3a_{21}x^2y+3a_{12}xy^2+a_{03}y^3)+\cdots,$$

où $a_{11}=D'/(AC)$, $a_{21}=-2b'D'/(A^2C)$, $a_{12}=-2bD'/(C^2A)$. L'arête de Green a pour paramètres directeurs

$$l_1=\frac{A}{4D'}\left(\frac{\partial\log c'}{\partial v}+4b-2a'\right), \quad m_1=\frac{C}{4D'}\left(\frac{\partial\log c}{\partial u}+4b'-2a\right), \quad n_1=1.$$

La droite générale du faisceau canonique a pour paramètres directeurs (λ paramètre variable)

$$L=\lambda l_1+l(1-\lambda), \quad M=\lambda m_1+m(1-\lambda), \quad N=1.$$

La droite λ est normale métrique si $L=M=0$ ou

$$\text{(I)} \quad \begin{cases} \lambda\dfrac{\partial\log c'}{\partial v}-2(1-\lambda)\dfrac{\partial\log c}{\partial v}-2(a'-2b)=0, \\ \lambda\dfrac{\partial\log c}{\partial u}-2(1-\lambda)\dfrac{\partial\log c'}{\partial u}-2(a-2b')=0, \end{cases}$$

d'où résulte $(2-\lambda)\partial^2\log(c/c')/\partial u\partial v=0$. Pour $\lambda=2$, on a la normale de Fubini et les relations

$$\frac{\partial\log cc'}{\partial u}=a-2b', \quad \frac{\partial\log cc'}{\partial v}=a'-2b,$$
$$cc'=k\left\{\frac{D'^2}{AC\sin w}\right\}^4, \quad k=\text{constante},$$

et la surface générale dépend de 6 fonctions arbitraires d'un argument.

Pour $\lambda\neq 2$, on peut supposer $c=c'$ et l'on obtient, en adjoignant aux équations (I) celles de Gauss-Codazzi, un système aux dérivées partielles assez compliqué, qui, pour $\lambda\neq 0$, n'est pas complet et dont la solution générale dépend seulement de constantes arbitraires. En particulier, si les quantités $A=C$, w, D', $a=a'$, $b=b'$, c sont fonctions du seul argument $u+v$, la solution dépend de 5 constantes arbitraires.

Pour $\lambda=0$, la solution dépend de deux fonctions arbitraires d'un argument et si l'on appelle $1/r$ la torsion des asymptotiques, $1/\rho_1$, $1/\rho_2$ leurs courbures, on a la relation géométrique $\rho_1^3\rho_{1u}r_u=\rho_1^3\rho_{1u}r_u$. *B. Gambier* (Paris).

Grove, V. G. On the Darboux tangents. Bull. Amer. Math. Soc. **53**, 1186–1191 (1947).

Étant donné un point O_1 sur une courbe plane C, on rapporte cette courbe à un triangle de référence $O_1O_2O_3$ covariant à C, le côté O_1O_2 étant tangent à la courbe. On considère en O_2 l'involution de rayons doubles O_2O_1, O_2O_3; des rayons correspondants coupent C en P_1, P_2; la position limite T du point de rencontre de P_1P_2 avec O_1O_3 est appelée point caractéristique du second ordre de C en O_1 par rapport à $O_1O_2O_3$.

Une surface S étant rapportée à ses asymptotiques, les coordonnées x^i du point courant O_1 sont normalisées de façon à satisfaire au système

$$x_{uu}=\theta_u x_u+\beta x_v+px,$$
$$x_{vv}=\gamma x_u+\theta_v x_v+qx.$$

Le tétraèdre local de référence est $O_1O_2O_3O_4$ dont les sommets O_2, O_3, O_4 ont pour coordonnées x_u^i, x_v^i, x_{uv}^i, respectivement. Le point O_v étant pris quelconque sur O_2O_3, le plan π déterminé par O_1, O_4, O_v coupe S suivant une courbe C_v dont le point caractéristique par rapport à $O_1O_vO_4$ est T_v. Le lieu de ce point caractéristique est une cubique rationnelle Γ_3 qu'on appelle courbe caractéristique de S et qui se trouve ainsi définie projectivement. Les points d'inflexion de Γ_3 sont sur les tangentes de Darboux qui sont ainsi définies d'une façon nouvelle.

La seconde arête de Green est obtenue simplement en utilisant les points caractéristiques par rapport à $O_1O_2O_3$ des deux branches de la courbe d'intersection de S par son plan tangent en O_1. L'auteur retrouve encore sous une forme nouvelle la correspondance de Moutard, puis la normale projective. *M. Decuyper* (Lille).

***Behari, Ram. The Differential Geometry of Ruled Surfaces.** Lucknow University Studies, no. 18, 1946. v+94 pp.

Le sujet de cet ouvrage est l'une des parties les plus attrayantes de la géométrie différentielle. Dans trois chapitres préliminaires l'auteur rappelle les notions essentielles sur les surfaces réglées. Il insiste surtout sur les propriétés de la ligne de striction dont la considération est si importante dans maintes théories géométriques, et donne, en particulier, une construction très simple d'une surface réglée de ligne de striction donnée. Rapportant une surface réglée R à ses génératrices rectilignes et à leurs trajectoires orthogonales, il étudie les variations des courbures totale et moyenne de la surface le long des lignes coordonnées, et étend certains des résultats obtenus à la surface S la plus générale en substituant, aux génératrices rectilignes et leurs trajectoires orthogonales de R, les asymptotiques d'une famille de S et leurs trajectoires orthogonales.

Le chapitre IV traite des asymptotiques des surfaces réglées. Cet ancien thème est rajeuni par une méthode d'exposition personnelle. Les cas classiques où la détermination des asymptotiques se fait par quadratures sont retrouvés très simplement. La considération des quadriques osculatrices donne lieu à d'élégants développements. Le cas où les quadriques osculatrices sont des paraboloïdes hyperboliques équilatères est spécialement envisagé; de nombreuses propriétés des surfaces pour lesquelles il en est ainsi sont mises en évidence, et l'auteur donne, à ce sujet, une signification géométrique de la fonction de Laguerre,

$$L'=\frac{d}{ds}\left(\frac{1}{R}\right)+\frac{2}{\tau}\cdot\frac{1}{\gamma},$$

où R, τ, γ sont les rayons de courbure normale, de torsion et de courbure géodésique d'une courbe quelconque tracée sur une surface arbitraire.

Le chapitre V étudie la déformation des surfaces réglées, passe en revue les théorèmes d'O. Bonnet sur la déformation avec conservation des génératrices rectilignes ou de Beltrami sur l'applicabilité de deux surfaces réglées avec parallélisme des génératrices, traite de quelques déformations particulières où une courbe de la surface initiale doit jouir de certaines propriétés sur la surface déformée, et signale un nouvel invariant de déformation. Le problème de la déformation infiniment petite d'une surface réglée, qui est en générale l'objet d'un traitement special, est directement rattaché au problème de la déformation d'une surface quelconque tel qu'il est envisagé par Darboux.

Le chapitre VI est le plus étendu; le lecteur y trouvera un exposé condensé et très complet de la théorie générale des congruences rectilignes. Les cinq familles de surfaces réglées issues d'un rayon d'une congruence rectiligne trouvées par K. Ogura [Sci. Rep. Tôhoku Imp. Univ. Ser. I. **5**, 107–120 (1916)] sont introduites de façon simple et originale, et une comparaison est faite entre la théorie des congruences rectilignes d'après Sannia et la théorie des surfaces d'après Gauss. Le cas particulier des congruences normales est étudié en détail. Il donne à l'auteur l'occasion de faire une étude approfondie d'un invariant intégral découvert par É. Cartan [Bull. Soc. Math. France **24**, 140–177 (1896)], invariant dont il indique des propriétés nombreuses et variées, et qu'il rattache à la notion d'anormalité d'une congruence, introduite par Levi-Civita, dont il donne une signification géométrique. L'ouvrage se termine par des généralisations données à des théorèmes classiques de Malus-Dupin, Beltrami et Ribaucour.

Les théorèmes II et III du chapitre VI, pp. 66, 68, énoncés pour une congruence normale quelconque, ne sont en réalité valables que pour les congruences des normales aux surfaces à représentation sphérique isotherme. Mais cette légère inadvertance, provenant du fait que l'auteur applique, à une congruence normale de directrice quelconque, des formules qui ne sont vraies que si la directrice est la surface moyenne, n'enlève rien à l'intéret d'un ouvrage très attrayant et particulièrement susceptible d'éveiller des vocations de géomètres. *P. Vincensini* (Besançon).

Tuganov, N. G. On basic lines on surfaces. Doklady Akad. Nauk SSSR (N.S.) **58**, 1911–1914 (1947). (Russian)

In a previous paper with the same title [same Doklady (N.S.) **57**, 327–330 (1947); these Rev. **9**, 201] the author studied affine-basic lines on a surface which are somewhat generalized in the present paper. A curve on a surface is affine-basic with respect to a given family of curves if the tangents to the family, along the curve, are parallel to the same plane. Referring the surface to asymptotic parameters the author obtains the differential equation of such a curve in terms of the fundamental quantities of the surface and the equation of the family. The author considers the envelope of the osculating planes of all the affine-basic lines through a point and shows that it is a cone of the third class. Another problem treated is that of conjugate nets, and that of surfaces for which all the osculating planes of the affine-basic lines through a point meet in a common line. *M. S. Knebelman* (Pullman, Wash.).

Hopf, Eberhard. Closed surfaces without conjugate points. Proc. Nat. Acad. Sci. U. S. A. **34**, 47–51 (1948).

Let S be a two-dimensional Riemannian manifold of class C^3 which is complete and compact. It is proved that,

if no geodesic on S has on it two mutually conjugate points, then the total curvature (the integral of the Gaussian curvature over S) of S must be negative or zero and in the latter case the Gaussian curvature must vanish identically on S. The first part of the conclusion yields only known results, but the second part solves a problem proposed by Morse and Hedlund [Trans. Amer. Math. Soc. 51, 362–386 (1942); these Rev. 3, 309], for with the aid of the Gauss-Bonnet formula it follows that if S is the topological image of the torus or of the Klein bottle, S must be the one-to-one and isometric image of a Euclidean model of such a surface. Morse and Hedlund were able to obtain this conclusion only under the additional hypothesis that no focal points exist on S. The proof is brief. It involves the construction of a function associated with the solutions of the Jacobi equations and integration of this function in the phase space.

G. A. Hedlund (New Haven, Conn.).

* **Debever, Robert. Sur une classe d'espaces à connexion euclidienne.** Thesis, Université Libre de Bruxelles, 1947. 96 pp.

Two generalizations of Riemannian spaces which are well known are due to Finsler and to Cartan. Finsler took as element of length between two points of coordinates x^r and x^r+dx^r a function $ds=L(x^r, dx^r)$ which is homogeneous of the first degree in dx^r. Cartan took the element of area in a 3-dimensional space as given by a $ds_2=L(x^r, d\sigma^{rs})=L(x^r, q^{rs})du^1du^2$ and deduced a definition of length of vectors from it. The author in this thesis has considered the problem of basing a geometry on a ds_2 in a 4-dimensional space. Difficulties arise in that case which do not occur in the cases treated by Finsler and Cartan. These difficulties arise from the fact that the bivector q^{rs} will then satisfy the equation $q^{12}q^{34}+q^{13}q^{42}+q^{14}q^{23}=0$ so that conditions imposed upon any function $L(x, p^{rs})$ involving the bivector p^{rs} will also be satisfied by the function

$$L^*(x, p)=L(x, p)+\Lambda(x, p)(p^{12}p^{34}+p^{13}p^{42}+p^{14}p^{23}),$$

where $\Lambda(x, p)$ is an arbitrary function of (x, p) of degree -1 in p. In order to determine the function $\Lambda(x, p)$ a supplementary condition is imposed, that the form giving the area shall be a form of rank 2. Similar indetermination occurs in the derivatives of L with respect to p^{rs}, and some further conditions have to be imposed in order to associate, in a unique way, a 4-index metric tensor to the given element of area.

Having introduced a bidimensional metric deducible from the fundamental function L the author considers a subclass of bidimensional metrics for which the osculating indicatrix $g_{rs,\,tu}X^{rs}X^{tu}=0$ is formed of biplanes tangent to a cone. This subclass is called the metric class, and for this class 2-index metric coefficients exist which are related to the 4-index coefficients by the relations $g_{rs,\,tu}=g_{rt}g_{su}-g_{ru}g_{st}$. The rest of the thesis is concerned with the metric class.

In the second chapter the author shows how, given a priori a bidimensional area of the metric class, a space of 2-dimensional elements of contact with a Euclidean connection can be determined. The third chapter is devoted to a treatment of the problems of the first two chapters by regarding them as problems of the point equivalence of manifolds of 4 dimensions in which the bidimensional area is given.

In the second part of the thesis all these results are generalized to the case of a manifold of n dimensions in which the m-dimensional area is given a priori. In this part there is a two-fold extension of the work of Kawaguchi and Hokari [Proc. Imp. Acad. Tokyo 16, 320–325 (1940); these Rev. 2, 167]. The class (called metric class in this thesis) of m-dimensional metrics from which a vector metric can be deduced is not specified by Kawaguchi and Hokari. Also the restrictions upon the values of m and n have been removed in this thesis.

E. T. Davies (Southampton).

Šul'man, T. A. On the bending of hypersurfaces in an affine space. Doklady Akad. Nauk SSSR (N.S.) 58, 1297–1299 (1947). (Russian)

At each point M of a hypersurface of $(n+1)$-dimensional affine space a moving polyhedral is determined by $n+1$ independent vectors $\mathbf{e}_i$, where $\mathbf{e}_1, \cdots, \mathbf{e}_n$ span the hyperplane osculating the hypersurface at M. The equations of infinitesimal displacement of such a polyhedral are $d\mathbf{M}=\omega^i\mathbf{e}_i$, $\omega^{n+1}=0$, $d\mathbf{e}_i=\omega_i^j\mathbf{e}_j$, $i, j=1, \cdots, n+1$. The vector $\mathbf{e}_{n+1}$ is selected so that (1) the volume of any n vectors in the osculating hyperplane is defined as the determinant of the coefficients expressing these vectors in terms of $\mathbf{e}_i$, (2) the projection of a nearby osculating plane on that at M parallel to $\mathbf{e}_{n+1}$ preserves the volume. These requirements on $\mathbf{e}_{n+1}$ being satisfied, the vector is called equiaffine and it determines an affine connection in the hyperspace. Two hypersurfaces are called applicable if a point correspondence between them can be established so that the affine connections determined by $\mathbf{e}_{n+1}$ coincide. The author proves that for $n>2$ the determination of a hypersurface applicable to a given one depends on one function of n parameters.

M. S. Knebelman (Pullman, Wash.).

Vagner, V. The geometrical theory of the simplest n-dimensional singular problem of the calculus of variations. Mat. Sbornik N.S. 21(63), 321–364 (1947). (Russian)

The paper is concerned with the variational problem $s=\int_{t_1}^{t_2}L(\xi^\alpha, \dot{\xi}^\alpha)dt$, where L is positively homogeneous of degree one in $\dot{\xi}^\alpha$ and the Hessian of L (with respect to $\dot{\xi}$) is of rank $n-1-r$; r is called the order of singularity. If ξ^α are considered to be coordinates in a space X_n, the equation $L(\xi, x)=1$ defines a hypersurface in the tangent centro-affine space E_n at the point ξ. This hypersurface is called the indicatrix of L. The hyperplanar coordinates of the indicatrix are given by $y_\alpha=L_\alpha(\xi, x)$ ($L_\alpha=\partial L(\xi, \dot{\xi})/\partial\dot{\xi}^\alpha$) and since the Jacobian of L_α is of rank $n-1-r$, y_α depend only on $n-1-r$ parameters; this means that the tangent hyperplanes to the indicatrix touch it in r-dimensional planes given by $L_\alpha(\xi, x)X^\alpha=1$ and $L_{\alpha\beta}(\xi, x)X^\alpha=0$. The rank of $\|L_\alpha, L_{\alpha\beta}\|$ being $n-r$, x^λ may be eliminated from $y_\alpha=\rho L_\alpha(\xi, x)$ giving $\underset{p}{h}(\xi, y)=0$, $p=1, \cdots, r$; $\rho=H(\xi, y)$, where the functions $\underset{p}{h}$, H are positively homogeneous of degree 1 in y^α, so that r-dimensional contact planes may be taken as $x^\alpha=H^\alpha+\lambda^p\underset{p}{h}^\alpha$ ($H^\alpha=\partial H/\partial y_\alpha$, $\underset{p}{h}^\alpha=\partial\underset{p}{h}/\partial y_\alpha$), λ^p being r parameters. The Euler equations of the problem

$$L_{\alpha\beta}\ddot{\xi}^\beta_\gamma+\partial_\gamma L_\alpha\dot{\xi}^\gamma-\partial_\alpha L=0$$

must be algebraically consistent, which requires that $\|L_{\alpha\beta}, \partial_\gamma L_\alpha\dot{\xi}^\gamma-\partial_\alpha L\|$ must be of rank $n-1-r$. This yields $(\underset{p}{h}, H)=\underset{p}{h}^\alpha\partial_\alpha H-H^\alpha\partial_\alpha\underset{p}{h}=0$ and $(\underset{p}{h}, \underset{q}{h})=0$, which enables us to write the Euler equations in canonical form $d\xi^\alpha/ds=H^\alpha(\xi, y)+\lambda^p\underset{p}{h}^\alpha$, $dy_\alpha/ds=-\partial_\alpha H-\lambda^p\partial_\alpha\underset{p}{h}$. Regarding λ^p as parameters, the integrals of these equations determine $(r+1)$-dimensional surfaces in X_n given by $\xi^\alpha=\Theta^\alpha(s, \lambda, \xi_0, \overset{0}{y})$ every curve of which is an extremal, the surfaces themselves being extremal surfaces for the problem. The main part of the paper narrows down these general

considerations to metrics of singularity order one. By means of the asymptotic cone to the indicatrix, which is assumed to be $(n-1)$-dimensional, the author is able to introduce a W-tensor density of $n-2$ dimensions and an invariant affine connection. Thus the differential geometry of singular metrics is developed, the fundamental theorem in this connection being: all differential invariants of locally singular hypersurfaces of the first order of singularity with an $(n-1)$-dimensional asymptotic cone for $n>3$ can be expressed in terms of the fundamental quantities g_{ab}, h_{ab}, v_{ab}, K and those obtained from them by the differential operators ∇_c, D^s, D_{n-1}, ϑ_n. For $n=3$ there are only three fundamental quantities $\mathfrak{B}$ and $\mathfrak{W}$. The last part of the paper deals with the first and second variation of a singular Finsler metric. The author obtains the necessary conditions of Jacobi and by introducing a convexity function δ, the necessary conditions of Weierstrass are obtained. *M. S. Knebelman.*

Galvani, Octave. La réalisation des connexions ponctuelles affines et la géométrie des groupes de Lie. J. Math. Pures Appl. (9) **25** (1946), 209–239 (1947).

L'auteur a montré ailleurs [Ann. Sci. École Norm. Sup. (3) **62**, 1–92 (1945); ces Rev. **7**, 530] comment on peut réaliser les espaces ponctuels à connexion euclidienne à torsion par des variétés d'éléments plans d'un espace euclidien. Dans ce travail, il continue ses recherches dans la même voie, en montrant qu'on peut réaliser un espace ponctuel à n dimensions à connexion affine à torsion par une variété à n dimensions d'éléments biplans plongée dans un espace affine à $n+q$ dimensions. Un élément biplan S_n^q d'un tel espace est défini par un point M et deux plans qui le contiennent, l'un $P(S)$ à n dimensions (la base), l'autre $Q(S)$ (le support) à q dimensions. La connexion se définit par la projection des points de la base de l'élément biplan S' voisin de S sur la base de S parallèlement au support de S.

Cette méthode est ensuite appliquée aux espaces de groupes de Cartan [J. Math. Pures Appl. (9) **6**, 1–119 (1927)] que l'auteur cherche à réaliser à l'aide d'une variété invariante par un groupe linéaire. Ce problème est résolu à l'aide d'un théorème d'Ado [Bull. Soc. Phys. Math. Kazan (3) **7**, 3–43 (1934); Cartan, J. Math. Pures Appl. (9) **17**, 1–12 (1938)] sur la représentation linéaire des groupes de Lie. Il est montré comment la résolution directe du problème pourrait conduire à une nouvelle démonstration du théorème d'Ado et du troisième théorème fondamental de Lie.

M. Haimovici (Jassy).

Eisenhart, Luther Pfahler. Finsler spaces derived from Riemann spaces by contact transformations. Ann. of Math. (2) **49**, 227–254 (1948).

La métrique d'un espace de Riemann étant définie par le tenseur $\gamma_{\alpha\beta}$, l'auteur applique à la forme $\Pi(\xi, \pi)=\frac{1}{2}\gamma^{\alpha\beta}\pi_\alpha\pi_\beta$ une transformation de contact homogène. Soit $P(x, p)$ l'expression obtenue. En y remplaçant les p_i par x'^i donnés par les relations $x'^i=\partial P/\partial p_i$ on trouve une expression $F(x, x')$ homogène du deuxième degré en x', qui est interprétée comme provenant de la métrique d'un espace de Finsler. Un vecteur t^i de cet espace est défini par les relations $t^i=\tau^\alpha[\partial x^i/\partial\xi^\alpha+\beta_{\alpha\gamma}\partial x^i/\partial\pi_\gamma]$ ou $\tau^\beta=t^i[\partial\xi^\beta/\partial x^i+b_{ij}\partial\xi^\beta/\partial p_j]$, τ^α étant les composantes de ce vecteur dans l'espace de Riemann et $\beta_{\alpha\gamma}=\pi_\beta\{{\beta \atop \alpha\gamma}\}$, $b_{ij}=x'^k[ij, k]+c_{ij}$ avec $c_{ij}=2G^k C_{ijk}$, $2G_h=x'^i\partial^3F/\partial x^i\partial x'^h\partial x^k-\partial F/\partial x^k$, $2C_{ijk}=\partial^3F/\partial x'^i\partial x'^j\partial x'^k$. L'auteur démontre ensuite que les géodésiques de l'espace de Riemann et celles de l'espace de Finsler se correspondent et qu'à la connexion de Levi-Civita du premier correspond celle de Cartan dans le second. *M. Haimovici* (Jassy).

Michal, Aristotle D. Infinite dimensional differential metrics with constant curvature. Proc. Nat. Acad. Sci. U. S. A. **34**, 17–21 (1948).

The author announces further results [cf. Bull. Amer. Math. Soc. **45**, 529–563 (1939); these Rev. **1**, 29] about infinitely dimensional spaces of constant curvature. To mention only one curious fact, for $k>0$ a closed geodesic retains its finite length $2\pi/\sqrt{k}$. The author also makes statements, to be published in full elsewhere, about infinitely dimensional Hermitian spaces, and he also justly notes that some of the concepts recently employed by the reviewer had been known to himself at an earlier date.

S. Bochner (Princeton, N. J.).

NUMERICAL AND GRAPHICAL METHODS

***Spenceley, G. W., and Spenceley, R. M. Smithsonian Elliptic Functions Tables.** Smithsonian Miscellaneous Collections, v. 109 (Publication 3863). The Smithsonian Institution, Washington, D. C., 1947. iv+366 pp. $4.50.

The book contains tables of functions sn u, cn u, dn u, $E(\phi, k)$, $A(r)$ and $D(r)$. The arguments θ and r take values $1°, 2°, \cdots, 90°$. The values of the functions have been computed with 15 decimal figures, the maximum divergence being ± 4 in the 15th decimal. In printing, they have been cut to 12 decimal figures; the tables assure the accuracy of the 12th decimal digit. The values of K, K', E, E', $D(90)$, $1/D(90)$ are given for each $\theta=1°, 2°, \cdots, 90°$ with 15 decimals, q and q' with 16 correct decimals. In the appendix there are short tables of hyperbolic functions with 15 decimals and of circular functions with 25 decimals for $r=1°, 2°, \cdots, 90°$.

P. J. Myrberg (Helsingfors).

***Moon, Parry. A Table of Planck's Function: 2000 to 3500°K.** Massachusetts Institute of Technology, Cambridge, Mass., 1947. 80 pp.

The tables give 8-figure values of

$$J_\lambda=\frac{C_1}{\lambda^5}\frac{1}{\exp(C_2/\lambda T)-1},$$

where λ=wavelength (micron), T=absolute temperature (°K), $C_1=36970$, $C_2=14320$, for $T=2000(10)3500$, $\lambda=0.38(0.01)0.76$. The first difference in the λ-direction is given; higher differences are, however, needed for interpolation to full accuracy. The tables supplement those given by the author [J. Math. Phys. Mass. Inst. Tech. **16**, 133–157 (1937); note error in this reference as given in the booklet under review]. The author notes that "The new table should be particularly useful, since it embraces the region of filament temperatures of incandescent lamps."

J. C. P. Miller (London).

Moon, Parry. A table of Planckian radiation. J. Opt. Soc. Amer. **38**, 291–294 (1948).
An extract from the book reviewed above.

Byhovskiĭ, M. L. The automatic calculating-analytical machine of Harvard University. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1947**, 1561–1575 (1947). (Russian)
An account based on the article by Aiken and Hopper, Elec. Engrg. **65**, 384–391, 449–454, 522–528 (1946); these Rev. **8**, 52.

***Svoboda, Antonín. Computing Mechanisms and Linkages.** Edited by Hubert M. James. Massachusetts Institute of Technology, Radiation Laboratory Series, vol. 25. New York and London, McGraw-Hill Book Company, Inc., 1948. xii+359 pp. $4.50.
This book is devoted to computers made entirely of mechanical elements. Such computers have been used for the rapid solution of gunfire control problems, in navigation and sometimes in other engineering applications. They are still the most economical, reliable and sturdy machines in those fields in which an accuracy of one part in a thousand is sufficient. A historical introduction describes the adding, multiplying, integrating, resolving and transforming components which have been customarily used in mechanical computers employing gears, cams, friction drives, worm drives and linkage mechanisms. But the principal concern of this book is a detailed study of the simplest of the mechanical methods, namely, the linkage mechanisms made entirely of rigid bars joined by pivots.
The design of linkage mechanisms has been an empirical art in which the principal techniques were graphical. The largest part of this book is devoted to an exposition of these graphical techniques of which very few have been published previously. Good design of linkage computers requires a proper compromise between the complexity of the mechanism to make it follow the variations of an analytical or an empirical function and the need for simplicity to reduce the accumulation of errors due to play in the hinges. The mathematical part of the problem is usually the determination of the parameters to allow the solution to pass through a sufficiently scattered set of arbitrary points on the desired curve with the hope that the derived curve does not depart too much at other points.
A harmonic transformer consists of a crank driving a sliding rod. By varying the lengths of the crank and rod, the pivot points and the initial points of the motions, one obtains an enormous selection of available transformations. Convenient tables are given in a 50-page appendix for the selection of the desired function. Combinations of two or more harmonic transformers give closer approximations. A study of the effect of structural errors on the final accuracy of the mechanism is included.
The three-bar linkage (more frequently called the four-bar linkage) is then investigated. This permits more types of curves and functions since there are more parameters. Numerical tables and graphical diagrams of families of curves and nomograms are given to enable first order design. Combinations of two and three harmonic transformers and a three-bar linkage are considered in several numerical examples.
Functions of two independent variables are mechanized approximately by means of linkages having two degrees of freedom. These are simpler and less expensive than the three-dimensional cams frequently used for this purpose.
Since the book is essentially a manual for designers, very few references are made to the literature. Important early work in the mathematical theory of approximating linkages was done by Burmester and Chebyshev. There has been much recent work by Z. Š. Bloh and N. G. Bruevič.
M. Goldberg (Washington, D. C.).

Storchi, Edoardo. Sulla quadratura approssimata del cerchio. Period. Mat. (4) **25**, 224–227 (1947).

Kolmogorov, A. N., Petrov, A. A., and Smirnov, Yu. M. A formula of Gauss in the theory of the method of least squares. Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 561–566 (1947). (Russian)
In articles 39–40 of Gauss's Theoria Combinationis Observationum Erroribus Minimis Obnoxiae there occurs the inequality $\rho\rho/\pi \leqq \sum(a\alpha+b\beta+c\gamma+\cdots)^2 < \pi$. Gauss failed to notice that this inequality can be sharpened. The purpose of the paper is to show that $\rho\rho/\pi \leqq \sum(a\alpha+b\beta+c\gamma+\cdots)^2 \leqq \rho$ and that this latter inequality cannot be improved.
W. E. Milne (Corvallis, Ore.).

Mal'cev, A. I. A remark on the work of A. N. Kolmogorov, A. A. Petrov, and Yu. M. Smirnov, "A formula of Gauss in the theory of least squares." Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 567–568 (1947). (Russian)
A question arising in the reasoning in the paper reviewed above was referred to Mal'cev who supplies the proof in this note. *W. E. Milne* (Corvallis, Ore.).

Piotrowski, S. L. Some remarks on the weights of unknowns as determined by the method of differential corrections. Proc. Nat. Acad. Sci. U. S. A. **34**, 23–26 (1948).
The method of least squares, when applied to nonlinear problems, is commonly carried out by means of differential corrections. In this paper it is shown that the mean errors of the differential corrections are equal to the mean errors of the unknowns only in certain special cases. In general the weights of the unknowns should be calculated from different formulas from those used for the weights of the differential corrections. *W. E. Milne* (Corvallis, Ore.).

Dupuy, Michel. Le calcul numérique des fonctions par l'interpolation barycentrique. C. R. Acad. Sci. Paris **226**, 158–159 (1948).
An adaptation of Lagrange's interpolation formula to machine calculation, with illustrative examples.
W. E. Milne (Corvallis, Ore.).

Pettit, John T. A speedy solution of the cubic. Math. Mag. **21**, 94–98 (1947).
The equation is reduced to $y^3+py-q=0$ or $y^3-py+q=0$, where $p, q>0$, and this by $y=qz/p$ to $z^3/(1-z)=\pm K$, where $K=p^3/q^2$. The paper gives a graph for (K, z) and a table with the interval 0.01 for the root Z. *E. Bodewig.*

Buchner, P. Das Hornersche Schema für komplexe Funktionswerte. Elemente der Math. **3**, 8–11 (1948).
This is, as the author states, an exposition of Collatz's well-known extension of Horner's scheme for complex arguments [Z. Angew. Math. Mech. **20**, 235–236 (1940); these Rev. **2**, 61]. Neither the author nor Collatz mention the outstanding feature of Horner's scheme, namely the reduction of the labor of calculation which, compared with the usual labor, is merely $\frac{1}{2}$ in the case of a real and even $\frac{1}{3}$ in

the case of a complex argument. When calculating the Newton correction $-f(a)/f'(a)$ the author commits the technical error of applying Horner's scheme to $f'(x)$ for computing $f'(a)$ instead of deriving $f'(a)$ from the continuation of Horner's scheme for $f(x)$ in a manner similar to that for a real argument. *E. Bodewig* (The Hague).

Wall, H. S. A modification of Newton's method. Amer. Math. Monthly **55**, 90–94 (1948).

The author rediscovers the formula $x_1=x_0-f/(f'-A)$, where $A=ff''/2f'$ and all functions are taken at the starting value $x=x_0$. Calling generally a sequence with the limit X "convergent of degree n" if $\lim[(x_{p+1}-X)/(x_p-X)^n]=c\neq 0$ it can easily be proved that for an analytic f the above formula gives a sequence of convergence 3, if the root to be approximated is simple, and of convergence 1, if the root is multiple, whereas Newton's original formula gives sequences of convergence 2 and 1, respectively. A similar formula giving convergence of degree n can easily be constructed. *E. Bodewig* (The Hague).

Bodewig, E. Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten. II, III. Nederl. Akad. Wetensch., Proc. **50**, 1104–1116, 1285–1295 = Indagationes Math. **9**, 518–530, 611–621 (1947).

Part I [same Proc. **50**, 930–941 = Indagationes Math. **9**, 441–452 (1947); these Rev. **9**, 250] gave an analysis of the number of operations required for the solution of simultaneous algebraic equations by Gauss's method, Cholesky's method and Schur's method. Parts II and III continue this analysis for the various methods associated with the names of Boltz, Banachiewicz, Jossa, Schmidt, von Mises, Pollaczek-Geiringer, Seidel, Morris, the relaxation method of Southwell, the iteration method of Cesari, etc. Part I dealt with direct solutions; part II, with methods that first find the inverse matrix; part III, with methods of iteration.

Each method presented is accompanied by a commentary and comparison with other similar methods. Samples. "The most practical of all methods is a modification of Gauss's method in which Gauss's algorithm is used iteratively." The method of Cholesky is applicable only to symmetric equations and is longer than that of Gauss but has the advantage of requiring only a small amount of written work. "If one has to solve k sets of n equations with the same matrix Gauss's method is better than Schur's if $k<n/3$. For $k>n/3$ the reverse is true." "The widespread opinion that Schmidt's method of orthogonalization is the most advantageous is shown to be false." "The normalization of a set of equations [in preparation for solution by iteration] requires for itself alone $1\frac{1}{2}$ times as many operations as the complete solution by Gauss's method. The normalization by the method of von Mises is hence utterly uneconomical, since after its completion one still has the solution of the new system to perform." "The relaxation method of Southwell is identical with the method of Seidel." Comparing the speed of convergence of the ordinary iteration, that of von Mises, and that of Seidel, he says, "The method of von Mises is the worst and that of Seidel the best." *W. E. Milne.*

Neville, E. H. Ill-conditioned sets of linear equations. Philos. Mag. (7) **39**, 35–48 (1948).

The author objects to the application of Morris' escalator process for solving ordinary systems of linear equations (not frequency-equations). Morris' solution of his example [same Mag. (7) **37**, 106–120 (1946); these Rev. **8**, 287] is incorrect because he did not take into account the accumulation of errors. The author therefore recommends another iteration process which runs as follows. (a) The system of equations is divided into parts. In every part only a partial set of the unknowns is kept on the left side, the other unknowns being brought to the right. Every partial system is solved, and the approximate solution of every part is inserted cyclically into the other solutions, in the manner of Seidel's process. (b) The solution of every partial system with matrix A, say, is obtained by means of the reciprocal matrix $R=A^{-1}$, where R is obtained by the quadratically convergent process $R_{i+1}=R_i(2E-AR_i)$ [which, by the way, was first described by G. Schulz, Z. Angew. Math. Mech. **13**, 57–59 (1933)]. The author's method seems to the reviewer to be very complicated, the number of operations being very large. A repetition of Gauss' elimination process seems to be more appropriate. On the other hand the device (a) has already been recommended by Morris himself [Aeronaut. Res. Committee [London], Rep. no. 1711 (1935)]. *E. Bodewig* (The Hague).

***Morris, Joseph. The Escalator Method in Engineering Vibration Problems.** John Wiley & Sons, Inc., New York, N. Y., 1947. xv+270 pp. $4.50.

In this book the author gives an account of his researches in structural and vibrational engineering, especially in aircraft engineering. In the first chapters he develops the theory of elasticity emphasizing the general reciprocal theorems, the method of strain energy and least work. His own researches are laid down in the chapters on the Hardy Cross method of moment distribution, the iteration methods for the solution of statical and vibrational problems in systems with many members or degrees of freedom and the "escalator method" for the practical solution of Lagrange's vibrational equations. The last proceeds by successive introduction and elimination of the various generalized coordinates in any order. It has the advantage that the calculation can begin even when some of the stiffness or inertial coefficients are unknown. The author gives numerous examples, mostly of problems of vibration in aeronautical engineering. The oscillations of the engine crankshaft, the propeller and the wings, longitudinal, pitching and yawing oscillations of an aircraft are treated. *E. Bodewig.*

Litschauer, J. Koordinatenumformungen mit der Doppelrechenmaschine. Österreich. Ing.-Arch. **2**, 18–63 (1947).

Bell, William D. A simplified punch-card approach to the solution of the flutter determinant. J. Aeronaut. Sci. **15**, 121–122 (1948).

Golden, S. An asymptotic expression for the energy levels of the rigid asymmetric rotor. J. Chem. Phys. **16**, 78–86 (1948).

The characteristic values of the real symmetric matrix E' of order $2J+1$ with the nonvanishing elements

$$E'_{K,K}=K^2,\quad E'_{K,K+2}=\alpha[f(J,K+1)]^{\frac{1}{2}},$$
$$E'_{K,K+2}\to\tfrac{1}{2}\alpha J(J+1),\qquad J\gg K,$$

α = constant,

$$f(J,n)=\tfrac{1}{2}[J(J+1)-n(n+1)][J(J+1)-n(n-1)],$$

are considered in the limit for large J. In this limit the matrix E' has precisely the same arrangement and dependences as the Hill determinant which results when a Fourier expansion is introduced into the Mathieu equation, except

for the fact that the Hill determinant is infinite. By adjoining zero elements to the E matrix it is possible to regard these differences and also differences from the asymptotic values as a perturbation upon the Mathieu equation. The effects of these upon the characteristic values are calculated by perturbation theory. The agreement with numerical methods is excellent. *H. Feshbach* (Cambridge, Mass.).

Golden, S. An asymptotic expression for the energy levels of the asymmetric rotor. II. Centrifugal distortion correction. J. Chem. Phys. 16, 250–253 (1948).

The method of the preceding paper is applied to a matrix whose approximate ($J \gg K$) nonvanishing elements are

$$E'_{K,K} = \alpha_1 + \beta_1 K^2 + \gamma K_1^4,$$
$$E'_{K,K\pm 2} \cong \alpha_2 + \beta_2 (K^2 \pm 2K),$$
$$E'_{K,K\pm 4} \cong \alpha_3 + \beta_3 (K^2 \pm 4K).$$

H. Feshbach (Cambridge, Mass.).

***Pollak, L. W., and Heilfron, C. Harmonic analysis and synthesis schedules for three to one hundred equidistant values of empiric functions.** Department of Industry and Commerce, Meteorological Service. Geophysical Publications, Vol. I. Stationery Office, Dublin, 1947. xxxiii+118 pp. £2/2/0.

The chief part of the work consists of tables for facilitating the computation of the first coefficients p_1, q_1 of the Fourier series, namely tables for $\cos iz$ and $\sin iz$, where $z = 2\pi/n$, $i = 0, 1, \cdots, n-1$ for $n = 3, 4, \cdots, 100$ at five decimals. Then according to Bessel, $p_1 = (2/n)\sum_i u_i \cos iz$, $q_1 = (2/n)\sum_i u_i \sin iz$, where u_i are the equidistant empirical values. Other tables, such as for values of iz, follow. The introduction contains Bessel's formulae for the Fourier coefficients, the division of the ordinates into groups, Lamont's correction for noncyclic change and Weihrauch's formulae.

E. Bodewig (The Hague).

Hampl, Miloslav. The sum of developments in orthogonal functions in technical problems. Rozpravy II. Třídy České Akad. 55, no. 5, 13 pp. (1945). (Czech)

In technological investigations problems like the following arise frequently. Given a function $H(\varphi)$ whose Fourier expansion is $\sum_2^\infty h_n \cos n\varphi$, to find the functions

$$H_2(\varphi) = \sum_2^\infty \frac{h_n}{n^2-1} \cos n\varphi, \quad H_3(\varphi) = \sum_2^\infty \frac{h_n \sin n\varphi}{n(n^2-1)^2},$$

$$H_4(\varphi) = \sum \frac{h_n \cos n\varphi}{(n^2-1)^2}.$$

In order to avoid laborious summation of the Fourier series, the author computes H_3 from the differential equation $H_3^{V} + 2H_3''' + H_3' = H$, and thereupon $H_4 = H_3'$ and

$$H_2 = -(H_3''' + H_3').$$

A. Erdélyi (Edinburgh).

Babkin, B. N. Approximate solution of ordinary differential equations of any order by a method of successive approximations based on a theorem of S. A. Čaplygin on differential inequalities. Doklady Akad. Nauk SSSR (N.S.) 59, 419–422 (1948). (Russian)

The method consists in establishing an upper as well as a lower limit for the solution sought and the application of an iteration scheme similar to Picard's to narrow the "fork" between the two bounds. (1) Equations of the first order. Let $y' = f(x, y)$, in which f and f_y are continuous in D, containing the point $M(x_0, y_0)$. Introduce upper bounds $v(x)$ and the lower bounds $u(x)$ of the solution:

$$v(x) > v_1(x) > \cdots > v_n(x) > \cdots > y(x),$$
$$u(x) < u_1(x) < \cdots < u_n(x) < \cdots < y(x),$$

such that $v' - f(x, y) > 0$, $u' - f(x, y) < 0$ for the interval $x_0 < x < x_1$; let $v(x_0) = u(x_0) = y_0$. Moreover, let $f_y > 0$ in the region D limited by the curves $y = v(x)$, $y = u(x)$ and the lines $x = x_0$, $x = x_1$. The iteration is carried out by means of

$$v_n(x) = y_0 + \int_{x_0}^{x} f(x, v_{n-1})dx,$$

$$u_n(x) = y_0 + \int_{x_0}^{x} f(x, u_{n-1})dx;$$

it is proved that $\lim_{n\to\infty} |v_n(x) - u_n(x)| = 0$ for $x_0 < x < x_1$.

(2) Equations of higher order. Let

$$(*) \qquad y'' = f(x, y, y'),$$

$f, f_y, f_{y'}$ being continuous in D, containing the point $M(x_0, y_0)$. Moreover, $f_y > 0$, $f_{y'} > 0$ in a finite interval (A, B). Call $y(x_0) = y_0$; $y'(x_0) = y_0'$ and the solution of (*) $y = y(x)$. Introduce $y = v(x)$ and $y = u(x)$ which for $(x_0, x) \varepsilon D$ satisfy

$$v'' - f(x, v, v') > 0; \quad u'' - f(x, u, u') < 0.$$

Call $v(x_0) = u(x_0) = y_0$, $v'(x_0) = u'(x_0) = y_0'$, $(v'(x))_{\max} = B$; $(u'(x))_{\min} = A$; then $v(x) > y(x)$; $u(x) < y(x)$. Calling $r_{n-1}(t) = v_{n-1}''(t) - f(t, v_{n-1}, v_{n-1}') > 0$,

$$s_{n-1}(t) = u_{n-1}''(t) - f(t, u_{n-1}, u_{n-1}') < 0,$$

the iteration formulas

$$v_n(x) = v_{n-1}(x) - \int_{x_0}^{x} (x-t) r_{n-1}(t)dt,$$

$$u_n(x) = u_{n-1}(x) - \int_{x_0}^{x} (x-t) s_{n-1}(t)dt$$

lead to the correct solution. The method can be extended to $y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$. *M. Daniloff.*

Hausman, L. F., and Schwarzschild, M. Automatic integration of linear sixth-order differential equations by means of punched-card machines. Rev. Sci. Instruments 18, 877–883 (1947).

The use of punched-card-relay calculators for the solution of systems of ordinary linear differential equations of total order six or less is described in considerable detail. Appropriate formulas are given for predicting and correcting the values for the $(n+1)$th step on the basis of the values up to the nth step. These formulas are automatically calculated by the machine. The method of putting the values of the coefficients and the starting values on the proper cards is explained. The problem of very large or very small values is discussed and appropriate transformations are suggested. Finally, the results in an actual numerical example are given. *W. E. Milne* (Corvallis, Ore.).

Zeilon, Nils. Some problems of numerical accuracy in the theory of differential and integral equations. Acta Univ. Lundensis [Lunds Univ. Årsskrift] N.S. Sect. 2. 43, no. 10=Acta Reg. Soc. Physiog. Lund. [Kungl. Fysiog. Sällskapets i Lund Handlingar] N.S. 58, no. 10, 30 pp. (1947).

In the first part of the paper the variational problem of the asymptotic phase of $u'' + \rho(x)u = 0$ is treated by approxi-

mating the differential equation itself. For the two cases of $\rho(x)=k^2+\rho_1(x)$, namely when $\rho_1(x)\sim ax^{-2}$ is a potential of Yukawa or similar type and $\rho_1(x)=-L(L+1)x^{-2}+be^{-x}x^{-1}$, i.e., the equation of the deuteron, the degree of accuracy attained is determined. The second part deals with the numerical computation of the characteristic numbers λ_i of an integral equation by iteration, thus avoiding linear equations. A method, similar to that of Trefftz, is described by which a very precise determination of λ_1 is obtained when a very rough limitation of $\lambda_2>\lambda_1$ is available. The method is that of approximating the first characteristic function of the problem. *E. Bodewig* (The Hague).

van de Hulst, H. C., and Reesinck, J. J. M. Line breadths and Voigt profiles. Astrophys. J. 106, 121–127 (1947).

The authors remark: "Voigt functions are useful approximations for the intensity distribution in spectral lines. A table of these functions is given, and its use in computations on instrumental distortion is explained. Some mathematical properties of the Voigt functions are mentioned."

Voigt functions [see also the following review] depend on two parameters β_1 and β_2 and it may be shown that, if $f'(x)$, $f''(x)$ are two such functions with parameters β_1', β_2' and β_1'', β_2'', then $f(x)=\int_{-\infty}^{\infty}f'(x-t)f''(t)dt$ is a Voigt function with parameters β_1, β_2, where $\beta_1=\beta_1'+\beta_1''$ and $\beta_2^2=\beta_2'^2+\beta_2''^2$. From these results, and from the limiting cases $\beta_1=0$, $f(x)=c_2e^{-x^2/\beta_2^2}$; $\beta_2=0$, $f(x)=c_1/(1+x^2/\beta_1^2)$, in which c_1 and c_2 are constants, it may be seen that the general Voigt function, with parameters β_1 and β_2, can be written

$$f(x)=c_1c_2\int_{-\infty}^{\infty}e^{-t^2/\beta_2^2}\left\{1+\frac{(x-t)^2}{\beta_1^2}\right\}^{-1}dt.$$

As indicated in "An Index of Mathematical Tables," by Fletcher, Miller and Rosenhead [Scientific Computing Service, London, 1946, p. 231, art. 15.85, 15.86; these Rev. 8, 286] this may be identified with

$$f(x)=\pi c_1c_2\beta_1\Re 2\pi^{-\frac{1}{2}}e^{z^2}\int_z^{\infty}e^{-t^2}dt,\quad z=(\beta_1+ix)/\beta_2,$$

that is, with the real part of a function associated with an error function of complex argument.

If h denotes the half-width, i.e., the value of x for which the ordinate $f(x)$ is half the central ordinate, the tables give two-decimal values of x/h for values 0.01, 0.02, 0.05, 0.1(0.1)0.8 of the ratio ordinate to central ordinate, and for values $\beta_1/h=0(0.025)0.5$. For each β_1/h, two-decimal values are also given for β_1/β_2, β_2/h, β_2^2/h^2, and p=area of curve/h×central ordinate; a graph of these is also given. *J. C. P. Miller* (London).

van de Hulst, H. C. Generalization of some methods for solving an integral equation of the first kind. Bull. Astr. Inst. Netherlands 10, 75–79 (1946).

The integral equation (1) $g(x)=\int_{-\infty}^{\infty}f(y)h(x-y)dy$ has important uses in spectral optics, mathematical statistics, and stellar statistics. The author calls $g(x)$ the observed distribution, $f(y)$ the true distribution and $h(x-y)$ the apparatus function. Equation (1) then shows how the true distribution f is modified by the observational apparatus to give the observed distribution g. An accurate solution of (1) is undesirable because of the fluctuations introduced in f due to random errors in g. What is required is a "stabilized" solution. The author discusses four methods of solution: (1) approximation by peaks, (2) Fourier expansion, (3) method of moments, (4) expansion in power series, and then concentrates on the method of power series which is worked out in considerable detail. *W. E. Milne* (Corvallis, Ore.).

Wittich, H. Konforme Abbildung einfach zusammenhängender Gebiete. Z. Angew. Math. Mech. 25/27, 131–132 (1947).

Discussion of several numerical methods.

Morgantini, Edmondo. Teoria dei nomogrammi a punti allineati ed a scale rettilinee, dal punto di vista delle corrispondenze plurilineari tra forme di prima specie. Rend. Sem. Mat. Univ. Padova 16, 3–72 (1947).

The object of this memoir is to give an exposition of the theory from an original and geometrical point of view. Chapter I sketches in projective terms the theory of the nomogram of genus zero. Chapter II treats the general alignment nomogram and extends it by replacing the straight transversals by a double infinity of curves. Chapter III treats the situation where two or more unknowns are determined as functions of two independent variables by the points in which a straight transversal cuts four or more straight supports. Chapter IV treats the case where one unknown is determined as a function of three variables by two straight transversals which cut four straight supports and meet on a straight hinge, and chapter V develops a theory of quadrilinear correspondence which gives a necessary and sufficient condition that an equation be representable by such a nomogram. *J. M. Thomas.*

Hruška, Václav. Note sur la construction des nomogrammes à points alignés. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 45 (1944), 249–262 (1945).

The nomogram is defined by a determinantal equation $|f_ig_ih_i|=0$ $(i=1, 2, 3)$, where f_i, g_i, h_i are respectively functions of z_i which can be represented in the neighborhood of each point ζ_i of the intervals I_i: $a_i\leqq z_i\leqq b_i$ in the form

$$f_i=|z_i-\zeta_i|^{p_i}f_i^*,\quad g_i=|z_i-\zeta_i|^{q_i}g_i^*,\quad h_i=|z_i-\zeta_i|^{r_i}h_i^*,$$

p_i, q_i, r_i being nonnegative real numbers and the functions f_i^*, g_i^*, h_i^* being continuous and having positive lower bound on $|z_i-\zeta_i|<\epsilon$. Except when $p_i\geqq r_i$, $q_i\geqq r_i$ for a fixed i, the point (x, y), where $x=f_i(\zeta_i)/h_i(\zeta_i)$, $y=g_i(\zeta_i)/h_i(\zeta_i)$, is at infinity. If the total number of points at infinity is finite and if the supports are curves which can be drawn, this paper shows that the scales can be fractioned into a finite number of parts so that the nomogram can be completely represented by a finite number of drawings, all of whose points are finite. *J. M. Thomas* (Durham, N. C.).

Hruška, Václav. Remarks on the construction of rectilinear nomograms. Rozpravy II. Třídy České Akad. 54, no. 21, 14 pp. (1944). (Czech)

A translation is reviewed above.

ASTRONOMY

de Jekhowsky, Benjamin. Réduction à sa plus simple expression du système principal d'équations des méthodes d'Olbers et de Gauss pour la détermination des orbites paraboliques. C. R. Acad. Sci. Paris **226**, 161–162 (1948).

The author shows that equations equivalent to those derived in a previous note [same C. R. **225**, 489–491 (1947); these Rev. **9**, 211] may be obtained from the two equations in the method of Lagrange-Andoyer. This leads to a slight modification in the form of the equations previously derived that render them more suitable for numerical applications.

D. Brouwer (New Haven, Conn.).

Buchanan, Daniel. Oscillating satellites with the force varying inversely as the nth power of the distance. Trans. Roy. Soc. Canada. Sect. III. (3) **41**, 27–43 (1947).

L'auteur a repris les recherches de F. R. Moulton [Periodic Orbits, Carnegie Institution, Washington, 1920, chap. 5] et T. Buck [ibid., chap. 8] sur les orbites périodiques du problème restreint des trois corps au voisinage des points de libration, avec une loi de force inversement proportionnelle à la n-ième puissance de la distance. L'auteur retrouve les trois points d'équilibre rectiligne et les deux points du triangle équilatéral. Au voisinage des points d'équilibre rectiligne, l'auteur construit, par analogie avec la méthode de Moulton, les trois classes A, B et C d'orbites périodiques à deux et trois dimensions. En suivant la méthode de Buck, l'auteur construit les orbites périodiques à deux et trois dimensions au voisinage des points d'équilibre du triangle équilatéral. Ces orbites existent seulement pour $0<n<3$. L'auteur ne donne pas une démonstration de la convergence des séries obtenues, mais il affirme que cette convergence peut être prouvée en utilisant l'extension du théorème de Cauchy par Poincaré et du théorème de MacMillan [Trans. Amer. Math. Soc. **13**, 146–158 (1914)]. *M. Kiveliovitch.*

Woltjer, J., Jr. On the anomalous phase-relation between first and second harmonic in the radial velocity of ζ Geminorum and related stars. Bull. Astr. Inst. Netherlands **9**, 435–440 (1943).

This continues earlier work [same Bull. no. 303]. Gas pressure is supposed small compared with radiation pressure; the star is supposed to have free periods approximately in the ratios $1:\frac{1}{2}:\frac{1}{3}$. The interaction of the corresponding oscillations in a finite motion is studied by using Hamiltonian equations of motion and introducing action and angle variables. A finite periodic motion exists in which all three oscillations are excited to determinate extents, and energy is generated by nonadiabatic influences in the first oscillation, transferred to the second and third, and dissipated in these. The damping and interaction terms are not determined from any assumed stellar model; values thought reasonable are, however, found by adjusting the theoretical radial-velocity curve to agree with that observed for ζ Geminorum. *T. G. Cowling* (Bangor).

Woltjer, J., Jr. On the possibility of a long-period secondary oscillation in ζ Geminorum. Bull. Astr. Inst. Netherlands **9**, 441–444 (1943).

The methods employed in the paper reviewed above are used to show that, when the periodic motion identified in that paper is excited, a slow secondary oscillation, also of definite size, may be superposed; its period is something like 10^5 times that of the main oscillation.

T. G. Cowling (Bangor).

Woltjer, J., Jr. On excitation and maintenance of secondary oscillations in pulsating stars. Bull. Astr. Inst. Netherlands **10**, 125–130 (1946).

The problems considered in the preceding two papers are rediscussed without any assumption as to the ratios of the free periods. The methods used are still those of generalized dynamics. The motion is supposed to depend on a number of independent periodic functions; the angle variable corresponding to the slowest free period is taken as independent variable. Successive approximations are introduced, involving increasing powers of the damping-constants. The general theory is applied to the special problem considered in the earlier papers. In addition to the very slow secondary oscillation found earlier, a less slow oscillation, 20 or so times as slow as the main one, is also identified: this is due to interaction between higher-order terms in the free oscillations. *T. G. Cowling* (Bangor).

Woltjer, J., Jr. On the theory of anadiabatic star-pulsations: a continuation, extended and emended. Bull. Astr. Inst. Netherlands **10**, 130–135 (1946).

This continues earlier work [same Bull. no. 282]. It determines how deviations from adiabatic conditions react on radial motions near the star's boundary and obtains a formula for the nonadiabatic part of the luminosity variation. For example, for a polytrope of order 3, the phase of this is $9\pi/10$ behind that of the adiabatic variation δL_r: its magnitude is

$$2^{-4/3}\frac{\Gamma(\frac{1}{3})\Gamma(1+4/(5\beta))}{\Gamma(\frac{4}{3}+4/(5\beta))}\left[-\xi^{3/2}dL_r/d\xi\right]_{\text{surface}},$$

where $(25/2)\xi^3=-(12\pi\rho T^2r^2/L_r)(dr/dT)(\gamma/\beta)[\partial U/\partial T]$, and U is the energy per unit mass. *T. G. Cowling.*

Woltjer, J., Jr. On an asymptotic relation between the variations of luminosity and radius in the theory of Cepheid-variables. Bull. Astr. Inst. Netherlands **10**, 135–136 (1946).

The asymptotic relation refers to the limiting case when gas pressure is small compared with radiation pressure. It connects the adiabatic variation δL in the surface luminosity L with the radial velocity V_r (positive if away from the observer): it is $L^{-1}\delta L=-(nk/g)dV_r/d\omega$, where ω is the phase, $2\pi/n$ the period, and k a positive constant of order unity. *T. G. Cowling* (Bangor).

Bhatnagar, P. L. The theory of anharmonic pulsations for cepheids: Two-phase homogeneous model. Proc. Nat. Inst. Sci. India **11**, 13–20 (1945).

Finite pulsations are considered for a model consisting of a uniform core surrounded by a uniform envelope. The relation between "skewness" in the velocity-time curve and amplitude of oscillation is shown to be independent of the disposition of mass inside the model. The period of small oscillations decreases with increasing condensation of the mass towards the centre. [It is in effect assumed that throughout the oscillation the displacement of any element is proportional to the distance from the centre; the actual

equation of motion is replaced by the energy equation. The results obtained are therefore only approximate.]

T. G. Cowling (Bangor).

Bhatnagar, P. L. Application of Rayleigh's principle to cepheid pulsation. Proc. Nat. Inst. Sci. India **11**, 25–29 (1945).

The general results obtained are equivalent to those of Ledoux and Pekeris [Astrophys. J. **94**, 124–135 (1941)].

In applications, the displacement of any element is assumed proportional to the distance from the centre.

T. G. Cowling (Bangor).

Moghe, D. N. On the theory of a spiral nebula. I. Proc. Nat. Inst. Sci. India **7**, 141–160 (1941).

Moghe, D. N. On the theory of a spiral nebula. II. Proc. Nat. Inst. Sci. India **7**, 161–175 (1941).

RELATIVITY

Fourès-Bruhat, Yvonne. Sur une expression intrinsèque du théorème de Gauss en relativité générale. C. R. Acad. Sci. Paris **226**, 218–220 (1948).

In his relativistic generalisation of Gauss's theorem, Lichnerowicz [Problèmes globaux de mécanique relativiste, Actual. Sci. Ind., no. 833, Hermann, Paris, 1939; these Rev. 1, 282] showed that the component R_0^0 of the Ricci tensor is the divergence of a space-vector h^λ. The author uses a moving rectangular frame of reference (repère mobile) in order to find an expression for the vector **h**, concluding with the remark that her equations show the existence of singularities in an exterior static "meublable" field and the locally Euclidean character of an external everywhere-regular field [cf. Lichnerowicz, same C. R. **222**, 432–434 (1946); these Rev. **7**, 397].

H. S. Ruse (Leeds).

Thiry, Yves. Les équations de la théorie unitaire de Kaluza. C. R. Acad. Sci. Paris **226**, 216–218 (1948).

It is shown that the equations of Kaluza's theory may be derived simply in terms of the exterior calculus of Cartan and Lichnerowicz, and that of the fifteen equations which so arise four may be interpreted as Maxwell's equations and ten as modified gravitational equations. It is stated that the physical significance of the extra equation will be examined.

H. C. Corben (Pittsburgh, Pa.).

Bergmann, Peter G. Unified field theory with fifteen field variables. Ann. of Math. (2) **49**, 255–264 (1948).

In this paper the author places on record part of a hitherto unpublished unified field theory due to P. Jordan. This theory is a generalization of Kaluza's five-dimensional unified field theory [see the author's book "Introduction to the Theory of Relativity," Prentice-Hall, New York, 1942; these Rev. **4**, 55, the notation of which is used in the present paper], and the essential idea is to do away with Kaluza's normalization condition. The effect is to add a new scalar function to the set of geometrical invariants. Physically, it may be interpreted as replacing the constant of gravitation by a secular variable, an idea which has appeared in other cosmological theories but is new to general relativity. It is remarked that the most serious drawback to Jordan's theory as it now stands is the multiplicity of possible Lagrangians. It is not at present known how Jordan intended to develop his theory, but there appears to be no logical reason for choosing one form of Lagrangian as against another.

A. G. Walker (Sheffield).

Schild, A. Discrete space-time and integral Lorentz transformations. Physical Rev. (2) **73**, 414–415 (1948).

Restricting the four coordinates in Minkowski space-time to take only integral values, it is shown that there exists an infinite group of Lorentz transformations with integral coefficients. It is thus possible to construct a model of discrete space-time which, although not completely Lorentz invariant, admits a significantly large subgroup of the Lorentz group of transformations. A more detailed account is to be published elsewhere.

A. G. Walker (Sheffield).

Durand, Émile. Sur la dynamique relativiste des milieux continus. J. Math. Pures Appl. (9) **25** (1946), 179–185 (1947).

The author claims to complete the theory of Costa de Beauregard [J. Math. Pures Appl. (9) **23**, 211–217 (1944); these Rev. **7**, 350] by establishing the equation of continuity (or conservation of mass) for a continuous medium in relativity. See the following review.

J. L. Synge.

Costa de Beauregard, Olivier. Dynamique relativiste des milieux continus. La variation de la masse propre en fonction du travail des forces superficielles. J. Math. Pures Appl. (9) **25** (1946), 187–207 (1947).

The author points out that Durand's proof [see the preceding review] of the equation of continuity (1) $\partial_i(\rho_0 V^i)=0$ assumes implicitly the constancy of the proper masses of the particles of the fluid and that this is an hypothesis which need not be accepted a priori. However, he admits that (1) is actually deducible from hypotheses laid down earlier by him [same J. (9) **23**, 211–217 (1944); these Rev. **7**, 350] and proceeds to reconsider these hypotheses. He claims that his theory, although developed for the special theory of relativity, can be extended without difficulty to the general theory. [This is not obvious to the reviewer, since vector integrands occur.] The medium consists essentially of a congruence of world lines defined by their unit tangent vectors V^i and an invariant proper density ρ_0. The dynamics of the medium is studied by considering the motion of a drop, i.e., a thin tube of world lines of normal 3-dimensional cross section (volume) δu_0. There is a force-density f^i; the basic equations of the earlier theory were (2) $f^i=\partial_j(\rho_0 V^i V^j)$ and (3) $V_i f^i=0$. From these (1) follows immediately. The formula $\partial_i V^i$ for the expansion of the drop is then developed. [This is well known; cf. A. J. McConnell, Ann. Mat. Pura Appl. (4) **6**, 207–231 (1929), in particular, p. 224; J. L. Synge, Trans. Roy. Soc. Canada. Sect. III. **28**, 127–171 (1934), in particular, p. 171.] Hence follows, as a natural definition of incompressibility, the equation $\partial_i V^i=0$, implying with (1) that the mass $\rho_0\delta u_0$ of the drop does not change. The author suggests as a modification of (1) the equation (4) $f^i+f_j V^i V^j=\partial_j(\rho_0 V^i V^j)$. [Actually this also implies (1).] However, (2) or (4) is an equation of motion involving only body force; surface force is now introduced. The 3-dimensional cross section of the world tube has a 2-dimensional boundary; its skew symmetric surface element is denoted by δs^{ij}. It is assumed that in a nonviscous fluid there is a surface force $\delta F^{ij}=-\omega\delta s^{ij}$ across this element, where ω is the pressure. From this

concept the author obtains the usual equations of motion of a perfect fluid, more commonly and easily obtained from the vanishing of the divergence of an energy tensor. These equations appear to be a major objective of the paper, because now (through the introduction of surface force) the equation of continuity (1) no longer holds. The author has in fact succeeded in giving a significant interpretation to the pressure, which hitherto has been a rather formal parameter in the energy tensor. The paper contains also thermodynamic considerations for a perfect gas and foundations of a general theory of surface forces. *J. L. Synge.*

Hoffmann, Banesh. The gravitational electromagnetic, and vector meson fields and the similarity geometry. Physical Rev. (2) 73, 30–35 (1948).

In a previous paper [Physical Rev. (2) 72, 458–465 (1947); these Rev. 9, 107] gravitational and vector meson fields were described in a unified theory using general projective geometry. In the present paper it is shown that electromagnetic fields can also be included in a unified theory, and that for this purpose "similarity geometry" can be used, this being the restricted conformal geometry on which the conformal factor is independent of the coordinates. It is shown that the similarity geometry associated with space-time can be conveniently described in terms of six coordinates $x^0, \cdots, x^5$, where $x^1, \cdots, x^4$ are the coordinates of space-time, x^0 is the gauge variable, and x^0, x^5 transform by $\bar{x}^0 = x^0 + N^{-1} \log k$, $\bar{x}^5 = k^{-2}x^5$, N being a fixed and k an arbitrary constant. A similarity tensor transforms in the usual way under transformations of these six coordinates.

The field theory is based upon a symmetric second order similarity tensor $S_{\sigma\tau}$ chosen for the sake of simplicity to satisfy certain restrictions ($S_{05}=0$, and S_{55}/S_{00} independent of $x^1, \cdots, x^4$). The projective tensor $\gamma_{\alpha\beta}$ of the previous theory, which gives rise to the gravitational metric tensor and the vector field of the meson theory, is identified with

$$\frac{S_{\alpha\beta}}{S_{00}} - \frac{S_{\alpha 5}S_{\beta 5}}{S_{00}S_{55}},$$

and the electromagnetic vector-potential, ψ_α, is identified with $S_{\alpha 5}(S_{00}S_{55})^{-\frac{1}{2}}$. The field equations of general relativity, together with Maxwell's equations and the vector meson field equations, are seen to emerge from a simple variational principle based on the curvature scalar of $S_{\sigma\tau}$, the total Lagrangian appearing naturally as the sum of the Lagrangians of the three constituent fields. *A. G. Walker.*

Mariani, Jean. Électromagnétisme et relativité. Le magnétisme terrestre comme conséquence de la relativité générale. Cahiers de Physique, no. 28, 23–54 (1945).

This is the first part of a paper containing a new attack on the problem of the unified field theory of gravitation and electricity. The author interprets the action of an electromagnetic field on an electrified particle as a rotation of the velocity world-vector of the particle, and shows that every fluid vortex motion, in which the vortex vector is a unique function of position, can be regarded as the motion of an electrified fluid in an electromagnetic field given by the vortex. To obtain a geometry of the electromagnetic field, a torsion (in Cartan's sense) as well as a metric must be associated with space-time. In the continuation of the paper the author proposes to obtain the field equations comprising both gravitation and electricity. Two interesting results follow from the theory: (i) to every mass density must be associated a charge density, and matter which classical theory regards as neutral can be considered neutral only in the mean; (ii) a rotating material sphere, regarded as neutral, induces a magnetic field, appreciable only for celestial bodies. *A. J. McConnell* (Dublin).

MECHANICS

Fleischel, Gaston. Généralisation de la formule de Willis pour les trains epicycloïdaux. C. R. Acad. Sci. Paris 226, 220–222 (1948).

The three ratios of the angular speeds of the three members of an epicyclic gear train are proportional to p, $(p-1)/p$, $1/(1-p)$. This relation holds equally as well for any three gears in more complicated gear trains. *M. Goldberg.*

Dimentberg, F. M. The finite displacement of a spatial four-linkage with cylindrical pairs and the case of passive constraints. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 593–602 (1947). (Russian)

A four-bar spatial linkage in which the links are free to slide along three of the hinges is movable with one or more degrees of freedom. The constant parameters of this system are the lengths of the links (a length is the perpendicular distance between the hinges in a link) and the twists (a twist is the angle between the hinges in a link). The variables are the angles between the links. The relation between the variables is derived in terms of the parameters by means of vectors and finally expressed in analytic form.

Under special conditions, as in the case of a Bennett linkage in which the opposite lengths are equal and the opposite twists are equal, but the hinges do not intersect, motion of the linkage is possible only when the lengths are proportional to the sines of the twists. In this case the bars of the linkage do not need to make use of the freedom to slide along the hinges. Such a condition is called a passive constraint. Other special cases in which passive constraints exist are the spherical linkage in which all the hinges intersect in a point, the plane linkage, and various degenerate cases such as the three-bar and the double two-bar.
M. Goldberg (Washington, D. C.).

*Beghin, Henri. **Cours de Mécanique. Tome I.** Édition provisoire polycopiée. Gauthier-Villars, Paris, 1948. vii+588+9 pp. 1800 francs.

This is a provisional mimeographed edition of a text used in the author's course at the École Polytechnique. The topics range from $F=ma$ to the generalized equations of Lagrange and Appell. Vector analysis is used from the start. The treatment of mechanics is concise, but clear and rigorous. About half of the illustrative problems deal with situations of technical interest. A limited number of exercises follow each of the eighteen chapters. The comprehensive discussion of systems of units in mechanics, with their relation to those in heat and electricity, is excellent. As a whole the book is a very good up to date introduction to classical mechanics. *P. Franklin* (Cambridge, Mass.).

Bloh, Z. Š. On the synthesis of four-link mechanisms. USSR Acad. Sci. Bull. Dept. Tech. Sci. [Izvestia Akad. Nauk SSSR] **1940**, no. 1, 47–54 (1940). (Russian)

Bloh, Z. Š. Application of connecting-rod curves to the solution of the problem of the synthesis of plane mechanisms. USSR Acad. Sci. Bull. Dept. Tech. Sci. **1940**, no. 7, 31–38 (1940). (Russian)

Bloh, Z. Š. The design of four-link mechanisms with bounded acceleration of the working link. USSR Acad. Sci. Bull. Dept. Tech. Sci. **1940**, no. 8, 27–30 (1940). (Russian)

Bloh, Z. Š. On the theory of conchoidal mechanisms. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1941**, no. 4, 101–106 (1941). (Russian)

Bloh, Z. Š. On the synthesis of crosshead mechanisms for rectilinear guidance. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1941**, no. 5, 99–104 (1941). (Russian)

Bloh, Z. Š. On the theory of the symmetric mechanisms of Čebyšev and their modifications. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1941**, no. 6, 49–62 (1941). (Russian)

Bloh, Z. Š. On the synthesis of mechanisms with rests. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1941**, no. 7–8, 89–98 (1941). (Russian)

Bloh, Z. Š. On an application of the connecting-rod curve of a slide-block mechanism. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1942**, no. 5–6, 77–80 (1942). (Russian)

These papers derive the analytical expressions for the curves traced by points on simple linkage mechanisms such as a sliding block driven by a crank through a connecting-rod, four-bar linkages, and combinations of two or more of them. They are used to approximate desired curves or straight lines. Numerical examples are given. [For more recent papers on the same subject by the author cf. the same Bull. Cl. Sci. Tech. **1946**, 683–696 (1946); Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 492–494 (1945); these Rev. **8**, 100; **7**, 340.] *M. Goldberg.*

Söchting, Fritz. Freie erzwungene gedämpfte Schwingungen mit nicht linearer Kennlinie eines Systems mit einem Freiheitsgrad. Österreich. Ing.-Arch. **1**, 382–389 (1947).

An approximation similar to the Rayleigh method is applied to the pendulum equation and some similar equations. The size of the error is checked by comparison with other solutions. *P. Franklin* (Cambridge, Mass.).

Hostinský, Bohuslav. Nouveaux problèmes relatifs à la résonance. Publ. Fac. Sci. Univ. Masaryk **1946**, no. 282, 28 pp. (1946). (Czech. French summary)

A linear oscillator (mass μ, period τ_1, $\nu=2\pi/\tau_1$), initially at rest in its equilibrium position, is set in motion by a collision at time $t=\tau$ with a mass μ_0 moving with velocity u_0 (and having kinetic energy K); the collision is repeated at intervals of τ so that the mass μ_0 has the same velocity at each encounter. If g_n is the increase of velocity of μ during the nth encounter, the displacement at any time t between the Nth and $(N+1)$th collision is $\sum_{n=1}^{N} g_n \nu^{-1} \sin \nu(t-n\tau)$. The author derives a linear difference equation of the third order with constant coefficients for g_n, hence finds an explicit expression for g_n which he is able completely to discuss. In particular, he shows that the energy of the oscillator approaches a limit E as N increases indefinitely. In general, $E/K=(\mu_0/\mu)(\sin \pi\tau/\tau_1)^{-2}$. In the case of resonance, $\tau=\tau_1$, this formula holds no longer: instead, $E=K$ in this case. The paper contains diagrams of the resonance curves (E/K as function of $\pi\tau/\tau_1$) for $\mu_0/\mu=1, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}$.

A. Erdélyi (Edinburgh).

Kazda, Josef. Sur quelques cas particuliers de la résonance. Publ. Fac. Sci. Univ. Masaryk **1946**, no. 283, 15 pp. (1946). (Czech. French summary)

Some details arising from the investigations of Hostinský [cf. the preceding review]. Curves (and description of their graphical construction) representing the displacement as function of time when $\tau/\tau_1=1$ (resonance), $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$.

A. Erdélyi (Edinburgh).

Haag, Jules. Sur les vibrations de certaines machines reposant sur des ressorts. J. Math. Pures Appl. (9) **25** (1946), 257–288 (1947).

The author considers forced vibrations of a general mechanical system which can be described in the following way. A massive table is supported by springs, and is itself the support of one or more subsystems. Each subsystem contains a body which can rotate about an axis (fixed with respect to the table) or can vibrate in two dimensions about a position of equilibrium. The subsystems are maintained in periodic motion by a motor, which is itself regarded as one of the subsystems. The paper deals chiefly with the resonance frequencies of the system and with the amplitudes of motion of the several parts. In particular, the author is concerned with the problem of designing the system so as to avoid the occurrence of resonance and the associated large motions. After a derivation of the differential equations of motion for the general case, there follows an extensive discussion of several relatively simple special cases which are of practical importance. This discussion includes numerical calculations which illustrate various parts of the theory. *L. A. MacColl* (New York, N. Y.).

Pröll, A. Der beschleunigte Aufstieg eines Strahlflugkörpers. Österreich. Ing.-Arch. **1**, 175–190 (1946).

Trabant, E. A. The Riemannian geometry of the symmetric top. Revista Ci., Lima **49**, 269–281 (1947).

The Riemannian space described by a symmetric gyroscope is investigated. As a necessary and sufficient condition for the configuration space to be an Einstein space with constant curvature, it is found that the two moments of inertia computed with respect to the principal axes should be equal. *G. Kron* (Schenectady, N. Y.).

Hamel, Georg. Über den allgemeinen schweren Kreisel. Z. Angew. Math. Mech. **25/27**, 159–160 (1947).

The author fills in some gaps in the treatment by Schiff and Stäckel [Stäckel, Math. Ann. **65**, 538–555 (1908)] of the vector equations of the unsymmetrical top, showing that a special case treated by them is impossible. [The formula for γ on line 21 seems to contain a misprint. The factor in front of the radical should be $1/\Delta$ instead of $\frac{1}{2}$.]

D. C. Lewis (College Park, Md.).

Arrighi, Gino. Sul giroscopio magnetico. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) **1**, 195–204 (1947).

The "magnetic gyroscope," studied in this paper, is essentially a symmetrical rigid body with a fixed point on the axis of symmetry and subjected to forces due to two dipoles

disposed as follows: one is fixed in the body along the axis of symmetry; the other is fixed in space in line with the fixed point of the body. The potential function due to these dipoles is expanded in a series of Legendre polynomials, but this seems to have little to do with the subsequent discussion of precession which is straightforward and leads to expected results. It is also to be expected (as verified by the author) that, when the poles of the spatially fixed dipole are very distant from the fixed point compared with the poles of the other dipole, the motion is like that of an ordinary gyroscope under the action of gravity.

D. C. Lewis (College Park, Md.).

Grioli, G. Moto attorno al baricentro di un giroscopio soggetto a forze di potenza nulla. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 439–463 (1947).

The problem is that of an axially symmetric rigid body free to move about its center of gravity when the body carries an electric charge distribution and is immersed in a uniform magnetic field. Another physical interpretation is also mentioned involving centrifugal force due to the earth's rotation. The equations of motion belong to a general type, indicated by the author, for which integration by quadratures is always possible. As in the case of an ordinary gyroscope, elliptic functions play a dominating role and similar results are obtained. *D. C. Lewis.*

Chazy, Jean. Sur les changements de variables canoniques. C. R. Acad. Sci. Paris **225**, 1041–1044 (1947).

A summary of several well-known theorems on contact transformations. *D. C. Lewis* (College Park, Md.).

Mineur, Henri. Réduction d'une forme quadratique dans le groupe canonique linéaire. C. R. Acad. Sci. Paris **225**, 1254–1256 (1947).

The author gives a canonical form for the quadratic terms in the Hamiltonian of a conservative holonomic system having multiple characteristic exponents. Actually, he mentions only linear systems, but his result is evidently applicable to any analytic system in the neighborhood of an equilibrium point as indicated above. *D. C. Lewis.*

Hydrodynamics, Aerodynamics, Acoustics

Pailloux, Henri. Sur les équations du mouvement des fluides parfaits. C. R. Acad. Sci. Paris **225**, 1122–1124 (1947).

A form of the Eulerian hydrodynamical equations is obtained from a set of identities and the continuity law. By slight changes of variables the equations are expressed in almost symmetrical form. An integral form of the Eulerian equations is also given in space-time.

W. J. Nemerever (Ann Arbor, Mich.).

Vazsonyi, Andrew. On the aerodynamic design of axial-flow compressors and turbines. J. Appl. Mech. **15**, 53–64 (1948).

The mapping of an arbitrary periodic cascade of airfoils onto the exterior of a circle is essentially the problem treated here. The technique uses successive mappings each of which is performed graphically. When the potential flow about many configurations must be computed, this method would appear to be efficient. *G. F. Carrier* (Providence, R. I.).

Byušgens, S. S. The critical surface of an adiabatic flow. Doklady Akad. Nauk SSSR (N.S.) **58**, 365–368 (1947). (Russian)

The equation of continuity of a stationary adiabatic flow is cast into the form $(v^{-2}-a^{-2})dA/ds-\operatorname{div}\mathbf{i}=0$, where $v\mathbf{i}$ is the velocity vector, v the velocity, a the velocity of sound, s the arc length of a streamline, $a^2=dp/d\rho$, $A=a^2/(k-1)$; density ρ and pressure p are related by $p/p_0=(\rho/\rho_0)^k$. Moreover, Euler's equation gives $vdv/ds+dA/ds=0$. Surfaces for which $\operatorname{div}\mathbf{i}=0$ are called minimal or critical surfaces; if $\operatorname{div}\mathbf{i}$ is zero at all points the congruence $\mathbf{i}$ is called minimal. Certain geometrical properties of such congruences and surfaces are derived. For instance, all points where $v^2=a^2$, $dA/ds\neq\infty$, lie on minimal surfaces. A minimal surface is the locus of points where either the velocity is equal to the velocity of sound or both these quantities have extreme values when changing along appropriate stream lines. In a plane adiabatic flow the velocity of the current can reach the velocity of sound only in the points of inflection of the orthogonal trajectories of the stream lines. Finally, a necessary and sufficient condition that an adiabatic flow be critical ($v=a$) is that the congruence of stream lines be a special minimal congruence, that is, a minimal congruence whose curvature vectors $d\mathbf{i}/ds$ form a gradient field.

D. J. Struik (Cambridge, Mass.).

Isaacs, Rufus. Recent progress in compressible fluid theory. Amer. Math. Monthly **55**, 140–144 (1948).

Hicks, Bruce L., Montgomery, Donald J., and Wasserman, Robert H. On the one-dimensional theory of steady compressible fluid flow in ducts with friction and heat addition. J. Appl. Phys. **18**, 891–902 (1947).

Steady flow of a compressible fluid through a duct with variable cross section with heat addition and against frictional forces is studied as a one-dimensional problem. The equations of motion and of state and the laws of conservation of energy and of mass are formulated in differential form to reduce the problem of determining the flow variables, once the addition of heat, frictional forces and variation of cross-sectional area is prescribed, to the integration of a nonlinear system of three ordinary differential equations in three unknowns. This system expresses the rates of change of the flow variables in terms of the rates of change along the duct of the quantities prescribed above. Provided $M\neq1$, initial values of the flow variables at a position in the duct determine a unique flow through the duct, but for $M=1$ the flow is not necessarily uniquely determined. A general discussion, based on this system, of the phenomenon of choking is given and several special flows are considered. *M. H. Martin* (College Park, Md.).

***Sauer, R. The Method of Characteristics for the One-Dimensional Unsteady Flow of a Gas.** The Graduate Division of Applied Mathematics, Brown University. Translation no. **A9-T-2.** i+26 pp. 1948.

The original appeared in Ing. Arch. **13**, 79–89 (1942); these Rev. **4**, 260.

Reutter, F. Über eine angenäherte quasilineare Potentialgleichung der ebenen kompressiblen Strömung und ihre mittels der Legendretransformation zu gewinnenden Lösungen. Z. Angew. Math. Mech. **25/27**, 156–157 (1947).

The author gives the differential equation for the velocity potential of two-dimensional transonic flows first derived

by von Kármán [J. Math. Phys. Mass. Inst. Tech. 26, 182–190 (1947); these Rev. 9, 217]. This equation is linearized by Legendre transformation into variables U, V, the components of the disturbance velocity. The undisturbed flow is in the direction of the x-axis. The resultant equation is that of F. Tricomi [Atti Accad. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) 14, 133–247 (1923)]. The solution of this equation can be written as an infinite series each term of which is the product of a hyperbolic cosine in V and a function in U, in fact, as pointed out by von Kármán, Bessel functions of order $\frac{1}{3}$. For flows over a body, the solution must have a singularity at the point corresponding to the conditions at infinity. The author suggests a scheme of approximate solution for a given profile without horizontal slope, symmetrical with respect to the x-axis, by satisfying the boundary conditions at only a finite number of points. *H. S. Tsien* (Cambridge, Mass.).

Giqueaux, Maurice. Sur la géométrie des écoulements permanents des fluides compressibles. C. R. Acad. Sci. Paris 226, 222–224 (1948).

The author obtains some local geometric properties of the velocity vector associated with a steady compressible flow of a fluid in three dimensions. It is assumed that at each point of the fluid a surface S exists which is orthogonal to the stream line through the point. A necessary and sufficient condition for this is that: (1) the fluid be irrotational, or (2) the fluid be rotational with plane or axial symmetry. By introducing a frame of rectangular Cartesian coordinates, which coincide locally with the tangent to the stream line and the principal directions on S, the author is able to express the components of the rate of strain tensor in terms of the magnitude of the velocity vector, the radius of curvature of the stream line and the principal curvatures of S. Expressing the equation of continuity in terms of these quantities and eliminating the density by use of the equation of motion, the following relation is obtained: $(1-M^2)\partial V/\partial x = V/R$. Here V represents the magnitude of the velocity vector, M the Mach number, x the displacement along a stream line and R is the mean curvature of S. Thus, if $M<1$, the concavity of S is in the same direction of the stream line as that for which V increases. Other results of similar nature are obtained. *N. Coburn.*

Christianovich, S. A. Approximate integration of the equations of a supersonic gas flow. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 215–222 (1947). (Russian. English summary)

The author uses a transformed form of the differential equations of a plane adiabatic supersonic gas flow which he has given several years ago in a paper unavailable to the reviewer [Trudy Central. Aero-Gidrodinam. Inst., no. 543 (1941)]. These equations read as follows: (1) $\varphi_\xi = -\chi^\lambda \psi_\eta$, $\varphi_\eta = +\chi^\lambda \psi_\eta$, (2) $2\xi = \sigma - \theta$, $2\eta = \sigma + \theta$, where φ is the velocity potential, ψ the stream-function, θ the inclination of the velocity vector and χ and σ are functions of the dimensionless speed variable λ which equals 1 for the critical speed. These functions are given by the formulas

$$(3)\qquad \chi = \frac{\lambda^2-1}{\nu-\lambda^2}, \quad \sigma = \sigma' + C, \quad \sigma' = \int_1^\lambda \left(\frac{\lambda^2-1}{1-\lambda^2/\nu}\right)^{\frac{1}{2}} \frac{d\lambda}{\lambda},$$

where $\nu = (\gamma+1)/(\gamma-1)$, γ being the exponent in the pressure-density relation, and C is a constant. If a solution of (1) is known, the corresponding flow in the physical (x, y)-plane can be determined by quadratures.

The main idea of the present paper is to replace the relation $\chi = F(\sigma')$ given by (3) by another relation which permits one to obtain the general integral of (1) in closed form. The second order equations resulting from (1) may be reduced to the Euler equation $\psi_{\xi\eta} + [k/(\xi+\eta)](\psi_\xi + \psi_\eta)$ [cf. Darboux, Leçons sur la Théorie Générale des Surfaces, v. 2, 2d ed., Paris, 1915] if the (χ, σ') relation is of the form (4) $\chi^\lambda = A(\sigma' + C)^{2k}$. The author treats the cases $k=1$, $k=-1$. In the case $k=1$ the constants A, C may be chosen so as to obtain a good approximation to the actual (χ, σ') relation (3) for the range $1.06 < M < 2.2$ of the Mach number M. In case $k=-1$ a good approximation is obtained for $2.2 < M < 3.5$. If $k=1$ the general solution of (1) is given by $\psi = [f_1(\xi) + f_2(\eta)]/(\xi+\eta)$; for $k=-1$ the identical expression is obtained for φ. The functions f_1 and f_2 are in both cases arbitrary.

The author then discusses how the formulas for the transition to the physical plane (involving the density-speed relation) must be modified for the case of the (χ, σ') relation (4), $k=\pm 1$. Finally, the general integral of the simplified equations (1) is used to obtain solutions of various boundary value problems in closed form. In these problems the gas flow is determined by the following data: values of φ and ψ on two characteristics, Cauchy data along a curve, values of φ and ψ along a characteristic and the presence of a free boundary, values of φ and ψ along a characteristic and the presence of a fixed boundary. *L. Bers* (Syracuse, N. Y.).

Frankl', F. I. On a family of particular solutions of the equation of Darboux-Tricomi and their application to the approximate calculation of the critical current in a given plane-parallel Laval nozzle. Doklady Akad. Nauk SSSR (N.S.) 56, 683–686 (1947). (Russian)

In a previous paper [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 9, 387–422 (1945); these Rev. 8, 416] the author showed that in the neighborhood of the sonic line the motion of a gas in a two-dimensional Laval nozzle may be described approximately by the Darboux-Tricomi equation (*) $\eta\psi_{\theta\theta} + \psi_{\eta\eta} = 0$. Here η is a certain function of the speed, θ the inclination of the velocity vector, and $\psi = \psi(\eta, \theta)$ the stream-function. In this paper the following solutions of (*) valid in both the elliptic and hyperbolic half-planes are discussed:

$$\psi_n = (\mu-\lambda)^{\alpha_n} F[-\alpha_n, \tfrac{3}{2}-\alpha_n, \tfrac{4}{3}-\alpha_n, \mu/(\mu-\lambda)], \qquad n = 0, 1, \cdots,$$

where μ, λ are the characteristic coordinates:

$$\mu, \lambda = \theta \pm \tfrac{2}{3}(-\eta)^{\frac{3}{2}}, \quad \alpha_n = (2n+1)/3,$$

and F denotes the hypergeometric function. Various formulas for ψ_n are given and it is shown that these functions are algebraic. [Tricomi already showed that for $n \equiv \pm 1 \pmod 3$ ψ_n is a polynomial in θ, η.] For $n \equiv 0 \pmod 3$ and $\lambda < 0 < \mu$, ψ_n has three real branches. The author announces another particular solution Ψ_t of (*) depending upon a real parameter t which was found by Falkovich. The formula for Ψ_t is too complicated to be reproduced here. A finite sum of the form $\psi = A\Psi_t + \sum B_n \psi_n$ represents a transonic flow through some Laval nozzle L. An approximate procedure for finding the flow through a given nozzle $\bar{L}$ slightly different from L is described briefly. *L. Bers.*

Falkovich, S. V. A class of Laval nozzles. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 223–230 (1947). (Russian. English summary)

The author treats the two-dimensional transonic flow in a Laval nozzle on the basis of the simplified equations

(1) $\varphi_\theta=-bs^{\frac{1}{2}}\psi_s$, $\varphi_s=bs^{\frac{1}{2}}\psi_\theta$, where φ is the velocity potential, ψ the stream-function, b a constant, s a certain function of speed such that $s=0$ for the sonic speed and $ds=-(1-M^2)^{\frac{1}{2}}dq/q$ (q the speed, M the Mach number) and θ the inclination of the velocity vector. To use (1) in the neighborhood of the sonic speed amounts to replacing Chaplygin's equation by the Tricomi equation. The density-speed relation leading to (1) is discussed. It is shown that (2) $\psi=A_0\psi_0+A_1\psi_1+A\theta$, where A_0, A_1, A are arbitrary constants and

$$\psi_0=\left(\frac{2s_0}{\theta^2+s^2+s_0^2}\right)^{1/6}\times F\left\{\frac{1}{12},\frac{7}{12},1,\frac{[\theta^2+(s-s_0)^2][\theta^2+(s+s_0)^2]}{(\theta^2+s^2+s_0^2)^2}\right\}\times\tan^{-1}\frac{2s_0\theta}{\theta^2+s^2-s_0^2}$$

(F being the hypergeometric function),

$$\psi_1=[\theta+(\theta^2+s^2)^{\frac{1}{2}}]^{\frac{1}{2}}+[\theta-(\theta^2+s^2)^{\frac{1}{2}}]^{\frac{1}{2}}$$

is a solution of the second order equation resulting from (1). This solution is valid in the subsonic domain. It can be continued analytically into the supersonic region and determines a flow in a Laval nozzle with a shockless transition through the speed of sound. The formulas obtained by analytic continuation of (2) describe the flow in the region bounded by the sonic line and two Mach lines. The flow may be continued downstream by the characteristics method.

L. Bers (Syracuse, N. Y.).

Oswatitsch, Kl., und Rothstein, W. Das Strömungsfeld in einer Lavaldüse. Jahrbuch 1942 der Deutschen Luftfahrtforschung, I95–I102 (1942).

The paper describes an approximate procedure for the computation of a potential gas flow in a Laval nozzle. In the two-dimensional case the boundary of the nozzle is given by $y=\pm f(x)$, $f(-x)=f(x)$. The velocity components are represented in the form $u=u_0+a_2y^2/2!+a_4y^4/4!+\cdots$, $v=b_1y+b_3y^3/3!+\cdots$, where u_0, a_2, a_4, b_1, b_3 are unknown functions of x. Substituting the expressions for u and v into the equations of gas dynamics the authors obtain ordinary differential equations for $u_0,\cdots,b_3$ which they solve by successive approximations. A similar procedure is used in the case of rotationally symmetrical flows, except that in this case only the first approximation is given. Numerical results are given for several special cases. *L. Bers.*

Lighthill, M. J. The hodograph transformation in transonic flow. I. Symmetrical channels. Proc. Roy. Soc. London. Ser. A. 191, 323–341 (1947).

The author studies the problem of finding the shape of a symmetrical nozzle with the velocity along the axis (x-axis) specified. The velocity along each streamline is assumed to increase steadily. The singularity at the sonic velocity and $\theta=0$ (θ denotes the inclination of the velocity with respect to the axis of the nozzle) is first studied in the physical plane by using a power series in y^2. In the hodograph plane, the two characteristics of the hodograph differential equation passing through the sonic point and $\theta=0$ are lines of branch points. The region between these lines is a region of triple-valuedness for the stream function ψ. Outside this region ψ is single-valued. There are also singularities at the sonic point and the point corresponding to the specified condition at the exit of the nozzle. The author then proposes to construct ψ in the hodograph plane by (i) a part singular at the sonic point, (ii) a part singular at the exit velocity and (iii) a finite sum regular throughout: $A\theta+\sum_1^n A_n\psi_n(\tau)\sin n\theta$, where τ is the square of the velocity and the ψ_n's are hypergeometric functions. The A's are fixed by the required approximation to the specified velocity distribution along the axis. This solution is single-valued, convergent and represents ψ except a region near the sonic point in the nozzle. For this excluded region, the author inverts the solution to obtain a power series in ψ for θ. This is shown to be convergent for the region of interest. The type of solution considered by the author gives a nozzle having an infinitely long supersonic part. *H. S. Tsien.*

Lighthill, M. J. The hodograph transformation in transonic flow. III. Flow round a body. Proc. Roy. Soc. London. Ser. A. 191, 352–369 (1947).

[For parts I and II cf. the preceding review, and same vol. 341–351 (1947); these Rev. 9, 350.] If an incompressible field of flow round a body has complex potential $w(z)$ which is an analytic function outside the body, if $dw/dz=\zeta=qe^{-i\theta}$ is the complex velocity, and if near a certain stagnation point

$$(1)\qquad w=\sum_{n=0}^{\infty}c_n\zeta^n=\sum_{n=0}^{\infty}c_nq^ne^{-in\theta},$$

and the velocity at infinity is taken as unity, the question is how to construct the stream function ψ for a compressible flow which (i) tends to the imaginary part of (1) as the maximum velocity $q_{\max}\to\infty$, (ii) has the singularity at $q=q_0$, the velocity at infinity, the same as the singularity at $q=1$ for (1). The author constructs such a solution by

$$(2)\qquad \psi=\Im\sum_{n=0}^{\infty}c_n\psi_n(\tau)f(n,\tau_0)e^{-in\theta},$$

where $\tau=q^2/q_{\max}^2$, $\tau_0=q_0^2/q_{\max}^2$, $\psi_n(\tau)$ are the hypergeometric functions, and $f(n,\tau_0)$ is chosen to satisfy the conditions (i) and (ii). The next question is how to continue analytically over the whole region, i.e., for $\tau>\tau_0$. The author accomplishes this by considering the function $\psi_n(\tau)f(n,\tau_0)$ as a function of the parameter n over the whole complex n-plane. Then $\psi_n(\tau)f(n,\tau_0)$ can be represented as an infinite series each term of which contains a pole of $\psi_n(\tau)f(n,\tau_0)$. Thus (2) is transformed into

$$(3)\qquad \psi=\Im\left[f(0,\tau_0)w(\varepsilon^{s-s_0-i\theta})+\sum_{m=1}^{\infty}r_me^{-ip_m\theta}\int_{\zeta_0}^{\varepsilon^{s-s_0-i\theta}}\zeta^{-p_m}dw(\zeta)\right],$$

where s is a function of τ given in part II of this series and r_m is the residue of $n^{-1}\psi_n(\tau)f(n,\tau_0)$ at the pole $n=p_m$ of $\psi_n(\tau)f(n,\tau_0)$. With (3) the physical coordinates can be worked out in a similar form of series. The author then gives the solution when w is given as a generalized Laurent expansion. The last solution can be applied to supersonic regions.

When circulation is absent, the simplest form for $f(n,\tau_0)$ is ε^{-ns_0}. Explicit results with this form of $f(n,\tau_0)$ are then given. For flows with circulation the choice of $f(n,\tau_0)$ is more restrictive, since both ψ and the physical coordinates must be single-valued functions of (τ,θ). The author states that it is likely that the solution worked out is the only solution. In the appendix the work of Tsien and Kuo [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 995 (1946); these Rev. 8, 237] is discussed. *H. S. Tsien.*

Laitone, E. V. The linearized subsonic and supersonic flow about inclined slender bodies of revolution. J. Aeronaut. Sci. 14, 631–642 (1947).

This paper presents approximate formulas for various flow quantities, such as the pressure, drag, lift, and moment coefficients of bodies of revolution. The derivations are based on the solution to the linearized equation for the velocity potential given by the von Kármán-Moore integral over a source distribution for supersonic flows and by a similar integral for subsonic flows. When the integral is expanded in a Taylor series in powers of the cylindrical coordinate r, and the solution fitted to the boundary condition, an approximate formula is obtained for the unknown source distribution in terms of the body shape, the degree of the approximation depending on the order of the error term in the Taylor expansion. Much of the paper is based on the author's observation that, for increasing Mach number M, it is possible for the neglected higher order terms in r, which contain M as parameter, to become comparable with terms retained in the approximation. His conclusion from this fact that other approximations, such as that of von Kármán-Moore, are always invalid for larger M does not seem justified, since for given M (however large) the error term can be made arbitrarily small if r is sufficiently small, that is, if the body is sufficiently slender. For the same reason, the author modifies his approximate formulas for $M>2^{\frac{1}{2}}$. These changes seem arbitrary to the reviewer, although the author justifies them in terms of the above Mach number effect. *D. Gilbarg* (Bloomington, Ind.).

***Taussky, Olga. A boundary value problem for a hyperbolic differential equation arising in the theory of the nonuniform supersonic motion of an aerofoil.** Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pp. 421–435. Interscience Publishers, Inc., New York, 1948. $5.50.

The author discusses the problem of the nonstationary supersonic motion of a two-dimensional airfoil. With the aid of the fundamental solution for the appropriate equation Possio obtained a formula which expresses the velocity as an integral, the integrand of which depends only on the variation of the angle of attack of the airfoil. In the present paper, the author derives the same formula with the aid of Hadamard's method for solving hyperbolic equations. This involves in particular adapting Hadamard's concept of the finite part of an integrand to the present problem. The procedure also establishes uniqueness of the solution.

P. A. Lagerstrom (Pasadena, Calif.).

Krasil'ščikova, E. A. The disturbed motion of the air caused by the vibration of a wing moving with supersonic velocity. Doklady Akad. Nauk SSSR (N.S.) 56, 571–574 (1947). (Russian)

Krasilschikova, E. A. Disturbed motion of air caused by vibration of a wing moving at supersonic speed. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 147–164 (1947). (Russian. English summary)

[The first paper is a summary of the second.] The author considers the problem of a vibrating thin wing which moves at supersonic speed in the x-direction. The velocity potential $\varphi=\varphi^{(0)}(x, y, z)+\varphi^{(1)}(x, y, z, t)$ is assumed to satisfy the linearized equation $(M_0^2-1)\varphi_{xx}-\varphi_{yy}-\varphi_{zz}-\varphi_{tt}+2M_0\varphi_{xt}=0$. The flow is assumed to be symmetric with respect to the (x, y)-plane. Behind the wing a discontinuity strip Σ in the (x, y)-plane is assumed along which the values of φ_z, $\varphi_z^{(0)}$ and $(\varphi_t^{(1)}-M_0\varphi_x^{(1)})$ are equal for $z\to+0$ and $z\to-0$. On the intersection S of the wing with the (x, y)-plane we have $\varphi_z^{(0)}=A_0(x, y)$, $\varphi_z^{(1)}=\Re A_1(x, y)e^{i\omega t}$, where $z=A_0(x, y)$ is the equation of the boundary of the wing, and $A_1(x, y)$ and ω are given quantities. Along the Mach cone we have $\varphi(x, y, z, t)=0$. On the domain $\sigma=\sigma(x, y)$ bounded by the Mach cone and the discontinuity strip we have $\varphi(x, y, 0, t)=0$. The author assumes a solution $\varphi^{(1)}$ of the form

$$\varphi^{(1)}=\Re F(x, y, z)e^{\beta z+i\omega t},$$

where F satisfies a linear partial differential equation and corresponding boundary conditions. She considers first a wing of infinite span in which case F satisfies $F_{xx}-F_{zz}-\lambda^2F=0$, which is solved by Riemann's method. She obtains for F an expression which in the special case $\lambda=0$ coincides with the formula obtained previously by Ackeret.

Using the results obtained for the plane problem she obtains by means of Fourier series the expression

$$F(x, y)=\pi^{-1}\int\int_{\sigma(x, y)}\frac{A(\xi, \eta)\cos\{\lambda[(x-\xi)^2-(y-\eta)^2]^{\frac{1}{2}}\}}{\sqrt{\{(x-\xi)^2-(y-\eta)^2\}}}d\eta d\xi$$

for a wing of finite span. If the leading edge of the wing is given by $x=\psi(y)$, the trailing edge is $x=\chi(y)$. The assumption is made that on the contour the values of $|\partial\psi/\partial y|$ and $|\partial\chi/\partial y|$ do not exceed the cotangent of the Mach angle. The solution of a finite wing with arbitrary leading edge is reduced to an integral equation

$$\int\int_{\sigma(\xi, \eta)}\frac{\theta(\xi', \eta')\cos\{\lambda[(\xi-\xi')^2-(\eta-\eta')^2]^{\frac{1}{2}}\}}{\sqrt{\{(\xi-\xi')^2-(\eta-\eta')^2\}}}d\eta' d\xi'=D(\xi, \eta),$$

where D is a known function.

Finally, the author gives the solution of the second basic problem in the theory of vibrating wings: assuming that the distribution of pressure is given over a wing, to determine the normal components of the velocity and the form of oscillation. The solution is given for a wing of arbitrary shape in the plane. If the frequency of oscillation is 0, the results hold for the case of steady motion; in this special case some of the author's results coincide with formulas of Prandl and Schlichting. In the introduction, the author gives a short historical note on the oscillating wing problem. She was apparently not aware of the work of Theodorsen, Küssner and Cicala, who considered the problem for which reference is made only to Sedov. *S. Bergman.*

Görtler, H. Einfluss einer schwachen Wandwelligkeit auf den Verlauf der laminaren Grenzschichten. I. Z. Angew. Math. Mech. 25/27, 233–244 (1947). (German. Russian summary)

The author investigates the influence of a small sinusoidal disturbance upon the laminar boundary layer. The disturbance is assumed to have constant amplitude in the direction of the mean flow and can be interpreted as a sinusoidal variation of the outer potential flow due to a slight waviness of the wall. The influence of this perturbation upon the mean velocity profile is investigated. The stream function ψ has the form

$$\psi=\psi_0(x, y)+\epsilon\{f_1(y)\cos x+g_1(y)\sin x\}$$

written in dimensionless variables: $x=(2\pi/\lambda)x^*$, etc.; λ denotes the wave length of the perturbation; f_1 and

g_1 satisfy the equations $U_0(y)f_1' - U_0'(y)f_1 + \sigma g_1''' = 1$, $U_0(y)g_1' - U_0'(y)g_1 - \sigma f_1''' = 0$, $\sigma = (\nu/U_\infty\delta_0)(\lambda/2\pi\delta_0)$, with the boundary conditions

$$f_1(0) = f_1'(0) = 0; \quad f_1'(\infty) = 1; \quad g_1(0) = g_1'(0) = g_1'(\infty) = 0.$$

The solution is obtained in series form in powers of y, i.e., starting at the wall, and another series from the free stream edge of the boundary layer.

The theory is applied to Blasius' solution of flat plate flow. The strong effect of the perturbation upon the velocity profile near the wall which can lead to separation is shown. The wall disturbance is damped out to a considerable degree when it reaches the edge of the boundary layer, as is to be expected. *H. W. Liepmann* (Pasadena, Calif.).

Görtler, H. Einfluss einer schwachen Wandwelligkeit auf den Verlauf der laminaren Grenzschichten. II. Z. Angew. Math. Mech. **28**, 13–22 (1948).

The theory developed in the first paper [reviewed above] is carried through for the case of small wave length of the perturbation, i.e., $\lambda/\delta \ll 1$. For this case the effect of the perturbation is restricted to a narrow zone near the wall where the Blasius solution is nearly linear. The equations are thus more easily solved and can be expressed as integrals of Hankel functions of order $\frac{2}{3}$. The effect of the perturbation is felt essentially in a thin sublayer of thickness $\delta_* = \delta_0(\lambda/L)^{\frac{1}{2}}$, where L denotes the length of the plate from the stagnation point and where $\lambda \ll L$. [Reviewer's remark: The interesting comparison of the above results with the stability theory of the laminar layer on the one hand, and Taylor's transition criterion for external turbulent motion on the other hand, are not mentioned by the author.]

H. W. Liepmann (Pasadena, Calif.).

Schuh, H. Über die Lösung der laminaren Grenzschichtgleichung an der ebenen Platte für Geschwindigkeits- und Temperaturfeld bei veränderlichen Stoffwerten und für das Diffusionsfeld bei höheren Konzentrationen. Z. Angew. Math. Mech. **25/27**, 54–60 (1947). (German. Russian summary)

The author uses an extension of the method of Piercy and Preston to solve the laminar boundary layer flow along a flat plate for variable material constants, i.e., ρ, μ and λ are assumed to vary across the layer. Solutions are obtained for the case of a very viscous liquid with Prandtl numbers of 12.5 and 100, respectively, and for air (Prandtl number 0.7). A diffusion problem is also considered. The method appears to require comparatively little labor.

H. W. Liepmann (Pasadena, Calif.).

Casal, Pierre. Dissipation de l'énergie en turbulence homogène. C. R. Acad. Sci. Paris **226**, 57–59 (1948).

Let $u_j(x_1, x_2, x_3, t)$, $j = 1, 2, 3$, be the three components of the velocity in a homogeneous turbulent flow of an incompressible viscous fluid; the author assumes that the u_j are random functions of (x_1, x_2, x_3, t) such that the mean values:

$$\overline{u_j(x_1, x_2, x_3, t)u_j(x_1', x_2', x_3', t)}$$

depend only on $x_1' - x_1$, $x_2' - x_2$, $x_3' - x_3$ and t; he gives for the mean values of the dissipation function W and the vorticity ξ the equation $\overline{W} = \mu\overline{\xi^2}$, this statistical relation holding for every x_1, x_2, x_3, t. *J. Kampé de Fériet* (Lille).

Squire, H. B. Reconsideration of the theory of free turbulence. Philos. Mag. (7) **39**, 1–20 (1948).

Quade, W. Zur Theorie der ebenen stetigen Gaswellen von endlicher Schwingungsweite. Z. Angew. Math. Mech. **25/27**, 215–232 (1947). (German. Russian summary)

The author derives the equation for plane gas waves of finite amplitude in terms of Lagrangian coordinates, using the longitudinal displacement of the gas particles as the dependent variable. The resulting equation, which is of the Monge-Ampère type, can then be easily integrated in closed form by means of the method of characteristics. The solution of the problem of plane waves propagating in a semi-infinite tube with a piston at the closed end is found for both isothermal and adiabatic changes of state. The author finds that the integral surface always has a return edge and that the appearance of the Riemann-Hugoniot phenomenon, i.e., the steepening of the wave front, depends on the initial conditions of the problem. *R. C. Roberts.*

Brinkley, Stuart R., Jr., and Kirkwood, John G. Theory of the propagation of shock waves from infinite cylinders of explosive. Physical Rev. (2) **72**, 1109–1113 (1947).

A detonation wave travels at constant finite velocity down an infinitely long cylindrical stick of explosive, setting up a shock wave which spreads out into the exterior fluid. The authors examine solutions of the type for which the Chapman-Jouguet condition is fulfilled at the detonation front. The solution is based on a theory proposed in an earlier paper [Brinkley and Kirkwood, Physical Rev. (2) **71**, 606–611 (1947); these Rev. **9**, 217] wherein the necessity of solving the hydrodynamical equations is avoided by imposing a similarity restraint of a plausible nature on the energy-time curve. The pressure jump behind the shock and the shock wave energy then satisfy a pair of ordinary differential equations with radial distance as independent variable. The solution of these equations can also be made to yield the meridian of the shock front surface. *D. P. Ling.*

Davila Cuevas, Rafael. General equations of variations of temperature in the atmosphere; secondary, dynamic and adiabatic processes in the variation. Revista Ci., Lima **49**, 247–267 (1947). (Spanish)

Levine, Harold, and Schwinger, Julian. On the radiation of sound from an unflanged circular pipe. Physical Rev. (2) **73**, 383–406 (1948).

The authors solve the problem of sound radiation from a semi-infinite unflanged circular pipe, assuming axially symmetric excitation. The solution found is valid in the wavelength range of dominant mode propagation in the pipe. The problem is formulated as an integral equation of the Wiener-Hopf type for the radial derivative of the velocity potential. The solution of this integral equation is found by the methods of Wiener and Hopf employing the Fourier transform in the complex domain. The factoring of the transform equation is accomplished with the aid of Cauchy's integral theorem. The factors which contain terms given by integrals in the complex domain are reduced to real integrals and then evaluated numerically. From this the reflection coefficient for the velocity potential within the pipe and the power gain function are evaluated numerically.

A. Heins (Pittsburgh, Pa.).

Rudnick, Isadore. The propagation of an acoustic wave along a boundary. J. Acoust. Soc. Amer. **19**, 348–356 (1947); errata, **20**, 149 (1948).

The author investigates the distribution of a sound field from a point source near an absorbing boundary separating

two different media of densities ρ_1, ρ_2 and propagation constants k_1 and k_2. A similar problem was treated by H. Stenzel [Ann. Physik (5) 43, 1–31 (1943); these Rev. 5, 249], by another method. The acoustic field as represented by the velocity potential φ resembles the Hertzian potential for the analogous electromagnetic case (magnetic field distribution). The velocity potential φ in either medium, satisfying the appropriate boundary conditions, is evaluated by the method used by K. A. Norton [Proc. I. R. E. 25, 1203–1236 (1937)], who treated the electromagnetic case. For large imaginary values of k_2 one approximates φ by the expression

$$\varphi_1=\frac{e^{ik_1r_1}}{r_1}+\frac{e^{ik_1r_2}}{r_2}\{(1-R_p)F+R_p\},$$

$$F=\left[1+2iw^{\frac{1}{2}}e^{-w}\int_{-iw^{\frac{1}{2}}}^{\infty}e^{-s^2}ds\right];$$

r_1, r_2 are the respective distances from the source and its image to the receiver, R_p is the reflection coefficient for plane waves and

$$w=i\frac{2k_1r_2\rho_1^2k_2^2}{(1-R_p)^2\rho_2^2k_1^2}(1-(k_1^2/k_2^2)\cos^2\delta),$$

δ being the incident angle. As $w\to 0$, $F\to 1$ and as $w\to\infty$, $F\to 0$. For large z and when r is much greater than the wave length of sound, $F\to 0$. When both sender and receiver are located at the boundary, $\delta=\pi/2$ (case of grazing incidence), $r_1=r_2$, $R_p=-1$ and $\varphi=2Fe^{ik_1r_1}/r_1$. The intensity of sound now depends on the factor F as well as the distance r. When $k_2\ll k_1$ and real (perfect reflecting boundary) and $z_r>0$, $F\cong 0$; thus in this case there is little intensity lost due to F. If $|k_2^2|\gg|k_1^2|$ and if k_2^2 is pure imaginary $w\cong w_0\cong k_1(2\pi\nu)^2r_1$ and F is complex. The intensity of the field now depends on the frequency and a change of phase occurs at the receiver for different frequencies. The field decreases very rapidly with frequency due to the factor F, which depends on both the frequency and the distance r_1. For large values of the frequency, F decreases more rapidly than for low frequencies, which accounts for the disappearance of higher harmonics of low frequency sounds at large distances from the source. Likewise, the phase depends on the frequency as well as r_1, and one must take this dependence into account when making velocity measurements. If k_2 is real, or its imaginary part is very small, F is complex and is represented by a combination of Fresnel integrals. For the case of an air-water boundary and source in air (the reverse is treated by Stenzel) w_0 depends linearly on the frequency and the distance r_1 and in the audible range $F\cong 1$ even for r_1 very large. When both the sender and the receiver are at the boundary, F is very small even for short distances and it passes through a minimum in this case. For negative values of w_0 the phase increases with distance and for positive values of w_0 the reverse is true. Finally, when both sender and receiver are above ground (the boundary) and in the first medium, the amplitude passes through a minimum as the receiver rises above the boundary. When the sender is on the boundary this minimum decreases in value as the frequency is increased and it occurs at lower heights. The phase lag from a free field wave is greater, the greater the frequency; it decreases with height. *N. Chako.*

Elasticity, Plasticity

Eckart, Carl. The thermodynamics of irreversible processes. IV. The theory of elasticity and anelasticity. Physical Rev. (2) 73, 373–382 (1948).

The purpose of this paper is to indicate that the classical theory of elasticity can be extended to include anelastic phenomena. The author states that, in order to do this, one must eliminate two false assumptions on which the traditional theory is based, namely the "principle of a constant relaxed state" and "the principle of relaxability in the large." The first asserts that, "if the strains of a solid object were to be completely relaxed at time t, the resulting state would be the same as if the strains had been relaxed at some other time t_0." The second principle asserts that "the strains in a solid object can be completely relaxed by removing all external forces." The author quite correctly points out that the strains in an optical blank cannot be relaxed in this way. In asserting that the traditional theory is based on this principle, he disregards, however, the extensive literature on initial stresses in elastic bodies [for a survey of this field see H. Reissner, Z. Angew. Math. Mech. 11, 1–8 (1931)]. The author states that relaxability in the large should be replaced by relaxability in the small and points out the analogy with Euclidean and Riemannian geometry. This remark is not new [see, for instance, the article by E. Trefftz in Handbuch der Physik, v. 6, Springer, Berlin, 1929, p. 64]. Using rectangular Cartesian coordinates x_i and adopting the Eulerian point of view, the author represents the line elements in the strained and relaxed states by $(dl)^2=\delta_{ij}dx_idx_j$ and $(d\lambda)^2=g_{ij}dx_idx_j$, where δ_{ij} is the Euclidean metric tensor and g_{ij} a tensor specifying the state of strain. If u_i is the instantaneous velocity of the particle with the coordinates x_i, the rate of change of the relaxed length is given by $D(d\lambda)^2/Dt=2a_{ij}dx_idx_j$, where the "anelasticity tensor" $a_{ij}=\frac{1}{2}\{\partial g_{ij}/\partial t+u_k\partial g_{ij}/\partial x_k+g_{ik}\partial u_k/\partial x_j+g_{kj}\partial u_k/\partial x_i\}$. The principle of a constant relaxed state would require that the relaxed length $d\lambda$ be independent of t, and hence that $a_{ij}=0$ at all times. In terms of the anelasticity tensor a_{ij} and the determinant g of the matrix g_{ij}, the divergence of the velocity field may be written as follows:

$$\partial u_i/\partial x_i=g^{ij}a_{ij}-\tfrac{1}{2}D(\log g)/Dt.$$

The equation of continuity (expressing the conservation of matter) then takes the form $D[\log(vg^{\frac{1}{2}})]/Dt=g^{ij}a_{ij}$, where v is the specific volume. If the principle of a constant relaxed state were adopted, the product $vg^{\frac{1}{2}}$ would have to remain constant for each material element. In the author's theory, however, v and g become independent. Treating the internal energy U as a function of the entropy η, the specific volume v and the strain metric g_{ij}, the author establishes the following expression of the first law of thermodynamics:

$$\rho D\eta/Dt+\partial(q_i/\theta)/\partial x_i=-(q_i/\theta^2)\partial\theta/\partial x_i+(\epsilon_{ij}/\delta)(s_{ij}+p\delta_{ij}-\tfrac{1}{2}g_{ik}\sigma_{kj}-\tfrac{1}{2}g_{jk}\sigma_{ki})+(\sigma_{ij}/\theta)(a_{ij}-\tfrac{1}{2}g_{ik}\omega_{kj}-\tfrac{1}{2}g_{jk}\omega_{ki}),$$

where ρ denotes the density, θ the temperature, q_i the heat flow, s_{ij} the stress tensor, $p=\partial U/\partial v$ the hydrostatic pressure, $\sigma_{ij}=-\rho(\partial U/\partial g_{ij}+\partial U/\partial g_{ji})$ the "elastic stress," $\epsilon_{ij}=\frac{1}{2}(\partial u_i/\partial x_j+\partial u_j/\partial x_i)$ is the velocity strain and $\omega_{ij}=\frac{1}{2}(\partial u_i/\partial x_j-\partial u_j/\partial x_i)$ the vorticity. The second law of thermodynamics asserts that the right-hand side of this equation must be positive. This condition is satisfied by setting $q_i=-k\partial\theta/\partial x_i$ $(k>0)$, $s_{ij}=-p\delta_{ij}+\frac{1}{2}g_{ik}\sigma_{kj}+\frac{1}{2}g_{jk}\sigma_{ki}+N_{ijkl}\epsilon_{kl}$, $a_{ij}=\frac{1}{2}g_{ik}\omega_{kj}+\frac{1}{2}g_{jk}\omega_{ki}+M_{ijkl}\sigma_{kl}$, where the forms $N_{ijkl}\epsilon_{ij}\epsilon_{kl}$ and $M_{ijkl}\sigma_{ij}\sigma_{kl}$ are positive definite. This set of

equations represents a visco-elastic material; the propagation of waves in a material of this type is discussed. It is not clear whether truly plastic behavior could be discussed along similar lines. *W. Prager* (Providence, R. I.).

Riz, P. M. The theory of elasticity for large deformations exceeding the limit of proportionality. Doklady Akad. Nauk SSSR (N.S.) **59**, 223–225 (1948). (Russian)

The author suggests that in view of recent progress in the theory of large elastic deformations an approximate theory of large plastic deformations might be developed by considering an essentially elastic material, the elastic constants of which change suddenly when a certain invariant of the strain tensor reaches a critical value. [In the realm of infinitesimal deformations, the analogous precedure is open to strong objections as was first recognized by E. Melan, Ing.-Arch. **9**, 116–126 (1938). The author does not indicate whether these difficulties can be resolved in the present case.] *W. Prager* (Providence, R. I.).

MacGregor, C. W., Coffin, L. F., Jr., and Fisher, J. C. Partially plastic thick-walled tubes. J. Franklin Inst. **245**, 135–158 (1948).

The paper contains an analysis of the partial yielding of thick-walled cylindrical tubes under the simultaneous action of internal and external pressure and axial tension or compression, provided that ratios between all these loads are kept constant during the loading process. The axial strain is assumed to be constant throughout the tube and the analysis is based on the following relation between the stress tensor σ_{ij} and the strain tensor ϵ_{ij}:

$$E\epsilon_{ij}=(1+\nu+\tfrac{1}{2}D)\sigma_{ij}-(\nu+\tfrac{1}{2}D)\sigma_{11}\delta_{ij}.$$

Here E denotes Young's modulus, ν Poisson's ratio, δ_{ij} is the Kronecker delta, and D is a function of the radius which must be determined so that the octahedral shearing stress is everywhere a given function of the octahedral shearing strain. For a given material, this function may be determined by a simple tension test. The basic equations of the problem are established for an arbitrary relation between octahedral shearing stress and strain; they are actually solved, however, only for the case of a perfectly plastic material. The results are compared to those obtained from several older theories, but the recent work of N. M. Belyaev and A. K. Sinickiĭ [USSR Acad. Sci. Bull. Dept. Tech. Sci. [Izvestia Akad. Nauk SSSR] **1938**, no. 2, 3–54; no. 6, 46–58] is not discussed. *W. Prager* (Providence, R. I.).

Swida, W. Verfahren zur Bestimmung der Tragwerkformänderungen im elastisch-plastischen Zustand. Z. Angew. Math. Mech. **25/27**, 168–169 (1947).

A number of unrelated results concerning the plastic bending of straight and curved beams are stated without proof. *W. Prager* (Providence, R. I.).

Cattaneo, Carlo. Teoria del contatto elastico in seconda approssimazione. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 504–512 (1947).

Jung, F. Der Culmannsche und der Mohrsche Kreis. Österreich. Ing.-Arch. **1**, 408–410 (1947).

Torre, C. Einfluss der mittleren Hauptnormalspannung auf die Fliess- und Bruchgrenze. Österreich. Ing.-Arch. **1**, 316–342 (1947).

Denoting the principal stresses by $\sigma_1 \geqq \sigma_2 \geqq \sigma_3$, the author expresses the intermediate principal stress in the form $\sigma_2=\alpha\sigma_1+\beta\sigma_3$, where $\alpha+\beta=1$. The plane $\sigma_2=\alpha\sigma_1+\beta\sigma_3$ in the σ_1, σ_2, σ_3 space is assumed to represent the type of loading ("Belastungsart"), all points in this plane representing states of stress of the same type. For a fixed type of stress, the Mohr circles which represent states of stress at the limit of yielding or rupture admit an envelope. The geometrical relations between these envelopes and the plane sections of the limit surface $F(\sigma_1, \sigma_2, \sigma_3)=0$ are investigated. Various yield and rupture hypotheses as well as empirical data are discussed. *W. Prager* (Providence, R. I.).

Narodeckiĭ, M. Z. On the strains in an inhomogeneous circular cylinder. Doklady Akad. Nauk SSSR (N.S.) **58**, 1305–1308 (1947). (Russian)

The note is concerned with explicit determination of stresses in the following plane problem of linear elasticity. A two-dimensional region is bounded by concentric circles L_2 and L_1 of radii R_2 and R_1 ($R_2>R_1$), respectively. The elastic constants in the region exterior to the circle L_1 differ from those of the interior region. The distribution of external forces is prescribed over L_2 and the displacements are assumed to be continuous across L_1. The solution is given with the aid of analytic functions of a complex variable, in the form of infinite series which converge rapidly in a specific illustrative example given by the author.
I. S. Sokolnikoff (Los Angeles, Calif.).

Hölder, Ernst. Stabknickung als funktionale Verzweigung und Stabilitätsproblem. Jahrbuch **1940** der Deutschen Luftfahrtforschung, I799–I819 (1940).

The paper is concerned with the nonlinear boundary value problem arising in the determination of the equilibrium configurations of an elastic bar which is subjected to (1) axial loads of given distribution and critical intensity, (2) additional small axial loads of given distribution and (3) small transverse loads of given distribution. If the intensities of the loads under (2) and (3) are denoted by p and q, respectively, and if r is a characteristic deflection, then $q=br(p-cr^2)$, to within higher order terms, b and c being constants. This relation characterizes the bifurcation at the buckling point $p=q=r=0$. *W. Prager.*

Šapiro, G. S. Axially symmetric deformations of an ellipsoid of revolution. Doklady Akad. Nauk SSSR (N.S.) **58**, 1309–1312 (1947). (Russian)

The note contains calculations of stresses in an elastic medium exterior to an ellipsoid of revolution, subjected to axially-symmetric deformations. *I. S. Sokolnikoff.*

Thorne, C. J. Square plates fixed at points. J. Appl. Mech. **15**, 73–79 (1948).

The author solves the equation (*) $D\nabla^4 w=q$ for thin square plates having a uniform load or a centrally placed point load and which have their edges fixed as to deflection or slope at a finite number of symmetrically placed boundary points. An exact solution of (*) is constructed so as to include a combination of linearly independent biharmonic symmetrical polynomials, and the parameters are determined by satisfying the conditions assigned at the fixed boundary points. No specifications are made at the boundary edge between the point supports and hence the solutions are not unique. *D. L. Holl* (Ames, Iowa).

Ling, Chih-Bing. On the stresses in a plate containing two circular holes. J. Appl. Phys. **19**, 77–82 (1948).

The author finds that, in a bipolar coordinate system, an expansion of the stress function in a product series expansion

allows him to obtain the plane stress pattern in an infinite plate containing two like holes. The coefficient determination is developed for an arbitrary stress field at infinity and numerical computations are carried out for certain cases.
G. F. Carrier (Providence, R. I.).

Pflüger, A. Halbscheibe mit Randglied. Ein Spannungsgleichnis zum Problem der tragenden Linie. Z. Angew. Math. Mech. **25/27**, 177–185 (1947). (German. Russian summary)

The author discusses the stress-field in the semi-infinite plane disk under uniform tensile stress, "reinforced" by a finite tensile member of higher modulus of elasticity and with negligible resistance against bending deformations. The connection between the two bodies is such that only shearing forces are transferred from one to the other. The author considers this system to be a suitable idealization of the upper part of a timber girder reinforced by a steel "fish-plate." The equations of the system are developed, and reduced to an integrodifferential equation for the force acting in the tensile member. It is shown that this integrodifferential equation is in formal agreement with that to which L. Prandtl reduced the "supporting line theory of the aerofoil." Hence well-known results of aerodynamics find direct application to a problem of elasticity. The stress distribution is evaluated and represented graphically. The "reinforcement" proves in most cases to diminish the actual strength of the system. *P. F. Neményi.*

Schubert, A. Die Beullast dünner Kreisringplatten, die am Aussen- und Innenrand gleichmässigen Druck erfahren. Z. Angew. Math. Mech. **25/27**, 123–124 (1947).

The stability of a thin elastic plate with concentric circular boundaries under uniform radial pressure is investigated. The displacement function is found in terms of Bessel functions and the critical load is associated with the roots of a transcendental equation. Some numerical results are computed. *G. F. Carrier* (Providence, R. I.).

Torre, C. Zur Beulung versteifter Rechteckplatten bei veränderlicher Randbelastung. Österreich. Ing.-Arch. **1**, 137–148 (1946).

This paper is concerned with the problem of the buckling loads of simply supported rectangular plates of uniform thickness. It is assumed that the plates are reinforced by stiffeners parallel to the edges of the plate and that the loads causing buckling are normal loads varying linearly along the edges. The method of solution is to assume a trigonometric double series for the deflection and to obtain equations for the coefficients of the series from the variational form of the problem. The general results are evaluated for the case of a plate with one stiffener in the direction of the thrust, which varies linearly from zero to its maximum value. Numerous references are included to earlier papers containing somewhat less general results of a similar nature.
E. Reissner (Cambridge, Mass.).

Reutter, F. Über die Stabilität dreischichtiger Stäbe und Platten, deren mittlere aus einem Leichtstoff bestehende Schicht einen in Dickenrichtung veränderlichen Elastizitätsmodul hat. I. Z. Angew. Math. Mech. **28**, 1–12 (1948).

This paper deals with some questions concerning the elastic stability of sandwich-type plates. The main results are for the cylindrical buckling of a homogeneous plate resting on an elastic foundation, under the assumption that the modulus of elasticity of the foundation decreases exponentially from its value on the surface of the foundation. Also discussed is the problem of the stability of a three-layer system according to the two-dimensional theory of elastic stability. No references are included to earlier published work on buckling of sandwich plates. *E. Reissner.*

Gol'denveĭzer, A. L., and Lur'e, A. I. On the mathematical theory of the equilibrium of elastic shells. (Survey of the work published in the USSR.) Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 565–592 (1947). (Russian)

This is a condensed survey of the research literature on the subject published in Russia during the past decade. Three distinct directions are discernible: (a) theoretical investigations based on the fundamental equations of the mathematical theory of elasticity, (b) work on stability and vibrations, (c) papers concerned with the engineering applications of the theory. This survey is concerned only with the first aspect. The development surveyed in this article falls into three categories: (a) formulation of the basic equations of the theory of thin shells, which extends the classical theory of Love with the aid of modern tools of differential geometry; (b) specialization of the general three-dimensional problem of the theory of elasticity to a two-dimensional one by introducing certain geometrical hypotheses and physical assumptions; papers in this category are concerned with the analysis of the nature of the simplifications and with the study of the magnitude of errors inherent in them; (c) integration of the equations formulated in category (a). This is accomplished by replacing the complete system of equations by special systems yielding the information about the "edge effect" and the behavior of membrane or momentless shells.

In addition to the account of the general investigations falling in these categories, the survey contains a résumé of several problems of integration of systems of equations associated with specific geometrical forms. These include spherical shells, conical shells and shells with vanishing Gaussian curvature. The survey concludes with a bibliography of 48 items. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Vlasov, V. Z. On two representations of the equations of a spherical shell. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 521–526 (1947). (Russian)

It is known that, under suitable geometrical hypotheses, the equations of the thin shell theory are reducible to a two-dimensional boundary value problem, the formulation of which is not unique. The author gives two formulations of the problem. One of these consists of a system of 3 differential equations in which the dependent variables are the dilatation, rotation with respect to the normal to the middle surface and displacement along the normal. The second formulation yields a single partial differential equation of the eighth order, on a certain function which, with the aid of the functions entering in the first formulation, can be computed by differentiation. *I. S. Sokolnikoff.*

Vekua, I. N. Certain fundamental questions of the theory of thin spherical shells. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 499–516 (1947). (Russian)

The paper is concerned with the problem of integration of the system of equations of equilibrium of a thin spherical

shell as formulated by Love. The formulas obtained give stresses, moments and displacements in terms of four arbitrary holomorphic functions. These results are applied to the shell in the form of a spherical segment whose edge is fixed, and to shallow shells for which the ratio of the height to the diameter of the base of the smallest spherical segment, containing the given shell in its interior, is small compared with unity. *I. S. Sokolnikoff.*

Ambarcumyan, S. A. On the calculation of shallow shells. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 527–532 (1947). (Russian)

The author investigates stresses in a shallow rectangular spherical shell with nonvanishing Gaussian curvature [cf. the preceding review] which is freely supported along the edge and subjected to normal loading. The computations are formal and include expressions for the transverse deflection of the shell. Numerical results are obtained for the case of a uniform normal load. *I. S. Sokolnikoff.*

White, Walter T. An integral-equation approach to problems of vibrating beams. II. J. Franklin Inst. 245, 117–133 (1948).

The investigation of part I [same J. 245, 25–36 (1948); these Rev. 9, 317] is continued. The Green's functions for twisted beams are developed in detail omitting any consideration of torsional displacements. Numerical results are quoted for a twisted turbine blade. *G. F. Carrier.*

Rollo, W. T., and Chambers, Ll. G. The vibrations of a pair of truncated cones placed base to base. Philos. Mag. (7) 38, 609–634 (1947).

Using the conventional one-dimensional beam vibration equation, the eigenfunctions and eigenvalues associated with the structure of the title are determined. The exact solutions of the equation are found in terms of Bessel functions and certain approximate solutions are also developed. The agreement is excellent. *G. F. Carrier* (Providence, R. I.).

Ludwig, Konrad. Die Querschwingungen der Schraubenfeder. Z. Angew. Math. Mech. 25/27, 29–31 (1947).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Ashford, Cyril. Parallax methods in optics. Philos. Mag. (7) 38, 580–592 (1947).

Looking through a lens at an object and a pin placed at an image point, parallax can only be avoided for all positions of the eye if the image is perfect. To evaluate the error, the author gives approximate formulae for the spherical aberration which he evaluates to compute parallax for different portions of the eye. With the help of this knowledge an experimental method is devised to locate the Gaussian focal point and the focal length of a lens. The amount of parallax and the error of the proposed method are discussed. *M. Herzberger* (Rochester, N. Y.).

Watzlawek, H. Sphärische Aberration dünner Einzellinsen nach D. Argentieri. Österreich. Ing.-Arch. 2, 114–122 (1948).

The author reports on some methods developed by Argentieri [Ottica Industriale, Milano, 1942] to treat the Seidel aberrations of thin lenses. The author uses as parameters two quantities ψ and μ, called the position factor and the form factor, respectively. If s, s' are the object and image distance, φ the power of the lens, then ψ and μ are defined by the equations

$$\varphi = s'^{-1} - s^{-1} = (n-1)(r_1^{-1} - r_2^{-1}), \quad \psi = s'^{-1} + s^{-1},$$
$$\mu = 2(n-1)^{-1}s'^{-1} - \psi n/(n-1) + 2/r_2.$$

The spherical aberration of a thin lens with these constants is given by $\varphi(A_0\mu^2 - 2\psi\mu + B_0^2\varphi^2)$, where $A_0 = (n+2)/n$, $B_0 = n/(n-1)$.

The author gives the solution for the thin biconvex lens with minimum spherical aberration for an infinite object as $r_1/r_2 = 1 - 2/\{(n-1)A_0^{-1} + B_0\}$ and compares Argentieri's parameters with those proposed by Clairaut-Schleiermacher, using $1/r$ and $1/s$, and Coddington-Taylor, using

$$\tau = (s^{-1} + s'^{-1})(s'^{-1} - s^{-1}), \quad \sigma = (r_1^{-1} + r_2^{-1})/(r_1^{-1} - r_2^{-1}).$$

[The r' used in section 6 of the paper is identical with r_2, the radius of the second surface.] *M. Herzberger.*

Brdička, Miroslav. Reflection of light by glass with a transparent homogeneous surface coating. Rozpravy II. Třídy České Akad. 57, no. 10, 18 pp. (1947). (Czech)

Arzelies, Henri. Sur l'intensité transmise par une lame transparente en régime de réflexion pseudo-totale. C. R. Acad. Sci. Paris 226, 478–480 (1948).

Searle, G. F. C. Interference fringes due to a wedge of air or liquid between two glass plates, in theory and practice. Philos. Mag. (7) 37, 361–389 (1946).

The paper develops the geometry of the set of interference fringes formed by light from a source of finite size incident on a wedge formed by partially reflecting and partly transmitting surfaces. The wedge angle is assumed small. The analysis includes rays with small inclinations to the principal planes of the dihedral angle. About half the paper deals with experimental techniques and results. *A. J. Kavanagh* (Buffalo, N. Y.).

Bloch, A. Inversion applied to the solution of 3-dimensional electromagnetic problems. J. Appl. Phys. 18, 1064–1071 (1947).

It is well known in what manner the action of a perfectly conducting sphere on the electric field of a point charge can be represented by a certain image charge. The underlying theory of inversion is extended to magnetic images. Whereas the magnetic scalar potential of a radial doublet, under inversion, does not behave in a simple manner, the magnetic vector potential does. This finds an immediate application to a current loop equidistant from a center. Simple laws of transformation are derived for magnetic flux through such loops and mutual inductance between them. The screening effect of spherical metal cans at high frequencies is studied. For instance, the mutual inductance between a sphere and a circular loop, equidistant from the center of the sphere, is identical to that between the original loop and its image in the sphere, and therefore easily expressed in terms of complete elliptic integrals.

The author emphasizes the incorrectness of earlier work of J. Larmor [Quart. J. Pure Appl. Math. 23, 94–101

(1889)] and of S. P. Thompson and M. Walker [Philos. Mag. (5) 39, 213–225 (1895)]. Reference is given to papers by H. Kaden [Elektr. Nachr. Tech. 10, 277–284 (1933)], C. F. Davidson and J. C. Simmonds [Wireless Engr. 22, 2–5 (1945)], and H. Buchholz [Arch. Electrotech. 28, 556–577 (1934)]. *C. J. Bouwkamp* (Eindhoven).

Franke, O. Über ein Extremalprinzip der elektromagnetischen Felder. Österreich. Ing.-Arch. 2, 89–93 (1948).

The author defines the "apparent voltage" u of an electrostatic field between two points A, B, as the line integral of the absolute value of the electric intensity along any path between A and B. Then the extremal principle of the title is simply that u is a minimum when the path from A to B is a line of force of the field (assuming A and B are on the same line of force). A similar integral can be used in the magnetic field, and these concepts are used to derive the tangent law of refraction at the interface between two media. *M. C. Gray* (New York, N. Y.).

Gans, R. La distribution du courant dans les antennes. Revue Sci. 85, 643–648 (1947).

An expository article on the Hallén integral equation for the current in a straight cylindrical antenna. Topics discussed include free vibrations, the short-circuited and loaded receiving antenna and the transmitting antenna. The author also suggests an entirely different method of approach based on Fourier series expansions, but does not give sufficient detail to enable the reviewer to judge its validity. *M. C. Gray* (New York, N. Y.).

King, Ronold, and Winternitz, T. W. The cylindrical antenna with gap. Quart. Appl. Math. 5, 403–416 (1948).

The authors extend the results of a paper by King and Middleton [same Quart. 3, 302–335 (1946); these Rev. 7, 401] to include the effect of a finite gap at the center of the antenna. A preliminary discussion shows that, if the antenna is center-driven from a two-wire line, the current distribution will not be appreciably affected except in the immediate neighborhood of the gap. Hence they modify the earlier results simply by integrating over the actual length of the antenna, excluding the gap $(-\delta, \delta)$. It is assumed that $\beta_0\delta \ll 1$ and a first order correction for finite δ is obtained. Curves showing the effect of the gap on the input impedance are included. *M. C. Gray* (New York, N. Y.).

Pistolkors, A. A. Theory of the circular diffraction antenna. Proc. I. R. E. 36, 56–60 (1948).

The circular diffraction antenna consists of a circular gap in an infinite perfectly conducting plane. An e.m.f. is applied across the gap and the field intensity at great distances is determined by the Fresnel diffraction theory. Vertical-plane radiation patterns of the antenna are given, showing that increasing the radius of the gap introduces additional lobes into the directive patterns. The author also obtains a formula for the current density on the conducting plane, from which the antenna admittance at the gap is determined. Assuming an infinitely narrow gap this consists of an infinite series corresponding to an infinite set of wave modes. The series for the conductance converges fairly rapidly and the total radiation conductance is shown to be similar to that of an oscillating sphere as obtained by Stratton and Chu [J. Appl. Phys. 12, 236–240 (1941)]. For the susceptance the series diverges and would have to be modified to include some conditions on the structure of the gap. *M. C. Gray* (New York, N. Y.).

Vol'pert, A. R., and Potehin, A. I. The diffractive field of a perfectly conducting sphere. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 17, 807–813 (1947). (Russian)

An exact solution to the problem of the diffraction of plane electromagnetic waves by a perfectly conducting sphere was given by G. Mie [Ann. Physik (4) 25, 377–445 (1908)]. The present paper adds, for the most part in graphical form, computational results for the phase and amplitude of the distant diffracted field in the directions respectively opposite to and coincident with the direction of propagation of the incident waves. *E. H. Linfoot.*

Saha, M. N., Banerjea, B. K., and Guha, U. C. On the propagation of e.m. waves through the upper atmosphere. Indian J. Phys. 21, 181–198 (1947).

The authors continue earlier work by Saha and collaborators [cf. M. N. Saha and B. K. Banerjea, same J. 9, 159–166 (1945); B. K. Banerjea, Proc. Roy. Soc. London. Ser. A. 190, 67–81 (1947); these Rev. 8, 186, 614] on the theory of propagation of electromagnetic waves through the ionosphere in the presence of the earth's magnetic field. The case of vertical propagation is studied in detail. Upon choosing the z-axis vertical, the (x, z)-plane parallel to the earth's magnetic field, the authors derive certain linear combinations of electric field components, viz., $E_x+iF_1E_y$ and $E_x+iF_2E_y$, which correspond to the ordinary and extraordinary waves, inasmuch as the atmosphere is assumed homogeneous. For an inhomogeneous atmosphere a coupling term arises; it vanishes on the magnetic equator and at the magnetic poles. Formulas are derived for the refractive indices of the two types of wave, the constants F_1 and F_2, the state of polarization of the waves, both with and without absorption. [There are several misprints; the caption to table I should read $g=-\omega \sin^2\theta/2 \cos\theta$.] *C. J. Bouwkamp* (Eindhoven).

Pekeris, C. L. The effect of ground constants on the characteristic values of the normal modes in non-standard propagation of microwaves. J. Appl. Phys. 19, 102–105 (1948).

Feuer, Paula, and Akeley, Edward S. Scattering of electromagnetic radiation by a thin circular ring in a circular wave guide. J. Appl. Phys. 19, 39–47 (1948).

Quantum Mechanics

Arnous, Edmond, et Massignon, Daniel. Grandeurs observables et fonctions aléatoires. C. R. Acad. Sci. Paris 226, 318–320 (1948).

This note attempts to find those "observables" (self-adjoint operators A in Hilbert space) which have the statistics of a stochastic process (fonction aléatoire). On account of the quantum interference of an observation with the state of a system, most observables will not be capable of being treated as stochastic processes. In order that this be possible, it is necessary and sufficient that, for any t_1, t_2 (epochs), the operators A_{t_1}, A_{t_2} commute. Here, A_t represents what becomes of the operator A in the course of the time t: $A_t = U_t^{\dagger} A U_t$, where U_t is the unitary operator governing the evolution of the system in time. *B. O. Koopman.*

Arnous, Edmond, et Massignon, Daniel. Fonctions aléatoires du second ordre et fonctions aléatoires simultanément enregistrables, en théorie quantique. Leurs représentations par des espaces de Hilbert. C. R. Acad. Sci. Paris 226, 557–559 (1948).

This note continues the ideas of the one reviewed above. It is desired to construct a Hilbert space representation in which each A_t is a vector, where A corresponds to a stochastic process. For this, it is necessary and sufficient that this stochastic process be of the second order (i.e., have finite first and second moments). The space is then taken as the set of all linear combinations with real coefficients of $A_{t_1}, A_{t_2}, \cdots$, the metric being defined by $(A_t, A_{t'}) = (A_t\psi, A_{t'}\psi)$, apparently for some particular ψ in the original Hilbert space (states of the system). The space for A is completed with the aid of the above metric. Generalizations, etc., are sketched out. *B. O. Koopman* (New York, N. Y.).

Blohincev, D. I. The principle of detailed balancing and quantum mechanics. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 17, 924–929 (1947). (Russian)

The principle of reversibility is a consequence of either classical or quantum mechanics. It can be stated as $P_{ik}(t) = P_{ki}(-t)$, expressing the equality of the probability of inverse transitions if the sign of the time is reversed. If k, l designate momenta and if the Hamiltonian is invariant with respect to a change of sign of k, l and the magnetic field H, one has the alternate form $P_{-l,-k}(t, -H) = P_{k,l}(t, H)$. In contrast, the principle of detailed balancing would be $P_{ik}(t) = P_{ki}(t)$. The method of perturbation with radiation damping is used to show that the principle of detailed balancing is not generally valid if the interaction forces are not central. This is demonstrated by the calculation of the scattering of a charged particle by a fixed dipole.

L. Tisza (Cambridge, Mass.).

Godner, I., and Sorokin, V. On the classification of functions of a system of identical particles by character of symmetry and moment of impulse. Doklady Akad. Nauk SSSR (N.S.) 58, 1931–1933 (1947). (Russian)

The linearly independent characteristic functions of n identical particles of angular momentum quantum number j span a space $(R_\alpha)^n$ with $\alpha = 2j+1$. Let u_α and $(u_\alpha)^n$ be the group of unitary rotations in R_α and $(R_\alpha)^n$, respectively. Then $(u_\alpha)^n = \sum_f a_f A(f_1, \cdots, f_\alpha)$, where $A(f_1, \cdots, f_\alpha)$ is the irreducible representation of u_α belonging to the partitio numerorum $f_1 + \cdots + f_\alpha = n$, $f_1 \geqq \cdots \geqq f_\alpha \geqq 0$. The rotations in ordinary space induce in $(R_\alpha)^n$ the rotations $(D_j)^n = \sum a_f A'(f_1, \cdots, f_\alpha)$. Here $A'(f_1, \cdots, f_\alpha) = \sum_J N_{(f)J} D_J$, where J is the total angular momentum quantum number; $N_{(f)J}$ has been determined to be the zero order term in the expansion according to ϵ of

$$\frac{(\epsilon^{2j+1}-1)(\epsilon-1)}{2\epsilon^{j+\alpha j+1}} \frac{D(\epsilon^{h_1}, \epsilon^{h_2}, \cdots, \epsilon^{h_\alpha})}{D(\epsilon^{\alpha-1}, \epsilon^{\alpha-2}, \cdots, 1)},$$

where $\epsilon = e^{i\varphi}$ (φ is the parameter of u_α), $h_k = f_k + \alpha - k$ and D is the Vandermonde determinant. *L. Tisza.*

Auluck, F. C. The artificially bounded relativistic linear oscillator. Proc. Nat. Inst. Sci. India 7, 383–391 (1941).

Davidson, P. M. The interaction of radiation and matter. Proc. Roy. Soc. London. Ser. A. 191, 542–552 (1947).

From the author's summary: "This paper is based on the idea that wave mechanics is primarily suitable for giving an indirect description of nature. The world to which the equations of the theory here proposed are directly applicable is an unnatural one, containing sources and sinks of matter which give to its radiational laws a symmetry lacking in nature. . . . The theory requires no representation of radiation other than the Maxwellian wave, and will be free from the difficulties encountered in other theories."

L. Hulthén (Lund).

Hamilton, J. The theory of radiation damping. Proc. Phys. Soc. 59, 917–940 (1947).

The quantum theory of radiation damping [Weisskopf and Wigner, Z. Physik 63, 54–73 (1930); 65, 18–29 (1930); Heitler, Proc. Cambridge Philos. Soc. 37, 291–300 (1941); Wilson, ibid. 37, 301–316 (1941); these Rev. 4, 95; Heitler and Peng, ibid. 38, 296–312 (1942)] is discussed in considerable mathematical detail for the system of radiation field and emitter or scatterer enclosed in a box. This facilitates calculation of the relation of degeneracy to divergence difficulties in radiation theory. The eigenvalue problem is chosen as that of a finite number of coupled oscillators.

For atomic emission the degenerate states do not enter into the solution. The eigenvalues are exhibited graphically. The result resembles that of Weisskopf and Wigner but includes the line shift indicated by Dirac [Z. Physik 44, 585–595 (1927)] and Oppenheimer [Physical Rev. (2) 35, 461–477 (1930)]. The connexion of this shift with the conservation of energy is investigated. Divergence difficulties for emission may, it is suggested, be removed by inclusion of physically unimportant higher order processes, e.g. atom jumping from normal to excited state with emission of photon. The scattering problem is simplified by idealised conditions of degeneracy and spacing of energy levels. A criterion for the validity or breakdown of perturbation theory is given. Again degeneracy has no crucial effect. For both problems various summations have to be cut off at some energy value which may be arbitrarily large.

C. Strachan (Aberdeen).

Balseiro, José A. Angular momentum of the radiation field. Revista Unión Mat. Argentina 12, 153–167, 209–224 (1947). (Spanish. English summary)

The theory of elementary particles permits one to attribute a well-defined angular momentum expression to a radiation field. Examples are given for the determination of the total angular momentum of an electromagnetic field of two photons and for the interferences characteristic of the Bose-Einstein statistics. Conservation of total angular momentum during an emission process leads to the expressions of the intensity distribution of the Zeeman effect in the limit $H \simeq 0$. In the case of a vector meson field total angular momentum can be split, in a relativistically invariant way, into orbital and spin momentum. The spin thus defined turns out to be an integral of motion for free particles.

Author's summary.

Schützer, Walter. Properties of Heisenberg's characteristic matrix for the one-body problem. Anais Acad. Brasil. Ci. 19, 283–297 (1947). (Portuguese. English summary)

The properties of Heisenberg's S matrix on principal axes are discussed in detail for five examples of motion of a particle in a potential field: the nonrelativistic and relativistic hydrogen atom, the potential well, the potential of Bethe-Ma ($-V_0 \cdot \exp(-r/a)$), and a potential well sur-

rounded by a potential barrier. The result of the calculations shows that only the two first problems present a simple behaviour of the characteristic matrix, in agreement with current views of the problem, while the three later examples show a great number of redundant singularities which are not related to closed states of the system considered.

Author's summary.

Schützer, Walter, and Tiomno, J. On the derivatives of the radiation field of the point electron with spin. Anais Acad. Brasil. Ci. **19**, 333–342 (1947). (Portuguese)

Schönberg, Mario. The Green functions of the Klein-Gordon equation. Revista Unión Mat. Argentina **12**, 238–264, 265–296 (1947). (Spanish. English summary)

The Green functions of the Klein-Gordon equation and other functions related to the Green functions, like the fundamental solution defined according to Hadamard's general method and the Riesz method of integration which correspond to the meson mass, are discussed, and considered as Green functions for complex integration. The solutions which describe the meson field with frequencies below that which correspond to the meson mass are discussed, and the consideration of these solutions allows one to describe satisfactorily the transition from a general meson field to a static one. This transition presents some new features in the case when the rest mass of the field quanta is not zero. Finally, the behaviour of the Green functions of the retarded and advanced fields in relation to the value of the variable in regard to the light cone is discussed. *Author's summary.*

McConnell, James. Production and annihilation of negative protons. II. Proc. Roy. Irish Acad. Sect. A. **51**, 173–190 (1947).

The calculations of paper I [same Proc. Sect. A. **50**, 189–221 (1945); these Rev. **7**, 272] are further elaborated. The cross-section for the production of pairs of positive and negative protons is calculated making use of the Møller-Rosenfeld form of the meson theory and the semiclassical Williams-Weizsäcker method. The cross-sections obtained in the present paper are somewhat smaller than those obtained previously. The detectability of negative protons in cosmic ray experiments is discussed. It is found that the probability of finding a negative proton is very small partly because of the small cross-section of production of proton pairs but also because of the rapid absorption of all fast nucleons owing to emission of mesons. *L. Jánossy.*

Le Couteur, K. J. Particles of half-odd integral spin. Proc. Cambridge Philos. Soc. **44**, 63–75 (1948).

Bhabha's theory of particles of higher spin is quantized for the case of half-odd spin. Although one is led to the possibility of negative probabilities for obtaining particular eigenvalues of a self-adjoint linear operator, it is possible to use the formalism to calculate transition probabilities. Developing the perturbation theory, the author applies the results of the scattering of mesons by particles which obey a particular equation of Bhabha's suggested as an equation for the proton, and by particles which in the nonrelativistic limit have a spin 3/2. In each case the tensor interaction is set equal to zero. The total cross-section for scattering by protons is found to be 17/9 times that calculated from the assumption that the proton obeys Dirac's equation.

H. C. Corben (Pittsburgh, Pa.).

Urban, Paul. Über die Entstehung und Vernichtung von Mesonen beim Durchgang durch Materie. Acta Physica Austriaca **1**, 55–73 (1947).

The cross-section for production of mesons by photon-nucleon collisions, and the inverse process, are calculated using vector theory and ordinary perturbation methods. The results are somewhat different from those of previous investigators [W. Heitler, Proc. Roy. Soc. London. Ser. A. **166**, 529–543 (1938); Kobayasi and Okayama, Proc. Phys.-Math. Soc. Japan (3) **21**, 1–13 (1939)] who are said to have overlooked a part of the matrix element which determines the transition probability. In consequence the ratio between the probabilities for meson production and electron-pair production, respectively, by high energy photons increases by a factor of 2. *L. Hulthén* (Lund).

Urban, Paul. Über die Entstehung von Mesonen durch Lichtquanten nach der skalaren Theorie. Acta Physica Austriaca **1**, 167–177 (1947).

The meson production by photon-nucleon collisions is treated on the basis of the scalar theory, using perturbation methods. The differential cross section, which shows a strong angular dependence for small energies, is integrated and the total cross-section obtained without specializing in low and high energies, as done in previous work [Yukawa, Sakata, Kobayasi and Taketani, Proc. Phys.-Math. Soc. Japan (3) **20**, 720–745 (1938); Nordheim and Nordheim, Phys. Rev. (2) **54**, 254–265 (1938)]. Discussing the earlier results, the author argues that the cross-section derived by Nordheim and Nordheim should be halved, if averaged correctly over the polarization directions of the incident photons.

L. Hulthén (Lund).

Petiau, Gérard. Sur l'approximation de l'optique géométrique dans la mécanique ondulatoire des corpuscules élémentaires. C. R. Acad. Sci. Paris **226**, 224–226 (1948).

Transition to the classical limit for the wave equation of particles with spin $nh/4\pi$ ($n=0, 1, 2, \cdots$). To each of the various mass states of the particle which occur for $n>1$ there corresponds a separate classical equation of motion corresponding to that envisaged mass. *A. Pais.*

Petiau, Gérard. Sur la force de Lorentz et la force de spin dans la théorie de l'électron de Dirac. C. R. Acad. Sci. Paris **226**, 313–314 (1948).

Petiau, Gérard. Sur la force de Lorentz et la force de spin dans la théorie du méson. C. R. Acad. Sci. Paris **226**, 387–388 (1948).

Petiau, Gérard. Sur l'approximation non relativiste de l'équation d'ondes du méson dans un champ de force central. J. Phys. Radium (8) **8**, 276–281 (1947).

Potier, Robert. Sur une théorie du méson à masses multiples. C. R. Acad. Sci. Paris **226**, 314–316 (1948).

Majumdar, R. C. The theory of pseudoscalar meson in presence of electromagnetic and nuclear fields. Proc. Nat. Inst. Sci. India **11**, 109–121 (1945).

Gupta, R. L. Sen. Specific ionisation of cosmic ray particles. Proc. Nat. Inst. Sci. India **9**, 295–300 (1 plate) (1943).

Korringa, J. On the calculation of the energy of a Bloch wave in a metal. Physica 13, 392–400 (1947).

Für die Berechnung der Energie $E(\mathbf{k})$ der Blochschen Welle (wo $\mathbf{k}$ den reduzierten Wellenvektor bedeutet) haben für $\mathbf{k}=0$ zuerst Wigner und Seitz [Physical Rev. (2) 43, 804–810 (1933); 46, 509–524 (1934)] eine Methode angegeben. Für grössere Werte von $\mathbf{k}$ hat diese Energie Slater [Physical Rev. (2) 45, 794–801] mit Hilfe einer Entwicklung nach Kugelflächenfunktionen berechnet. In der vorliegenden Arbeit wird die dynamische Theorie der Kristallgitterinterferenzen auf diesen Fall angewendet. Es sei $\varphi(\mathbf{r})$ die Lösung der Wellengleichung im Falle eines konstanten Potentials V_0, dann folgt für die vom Gitterpunkt (000) auslaufende Welle in erster Näherung $\psi_u=\varphi(\mathbf{r})$ und für die von allen anderen herrührenden einfallenden Wellen $\psi_v=\sum' \exp(-i\mathbf{k}\mathbf{r}_n)\varphi(\mathbf{r}-\mathbf{r}_n)$. Die Lösung der Wellengleichung wird nach Kasterin [Akad. Wetensch. Amsterdam. Verslagen Wis. Nat. Afd. 6, 460–480, 532 (1898)] folgendermassen angesetzt: $\psi=\psi_u+\psi_v$, $\psi_u=\sum_{lmj}a_{lmj}f_l(k_0r)Y_{lmj}(\vartheta,\chi)$ und $\psi_v=\sum_{lmj}b_{lmj}g_l(k_0r)Y_{lmj}(\vartheta,\chi)$, wo die Y Kugelflächenfunktionen und $f_l(k_0r)$ und $g_l(k_0r)$ Hankelsche bzw. Besselsche Funktionen bedeuten. Man erhält aber ψ_v als eine über alle Gitterpunkte erstreckte Summe der Funktionen vom Typ ψ_u und daraus folgt in der Nähe des Punktes $x=y=z=0$ eine Relation zwischen den a_{lmj} und b_{lmj}. Ein weiterer solcher Zusammenhang folgt aus der Zerstreuungstheorie, nach der $a_{lmj}=\frac{1}{2}b_{lmj}(1-e^{-2i\eta_l})$ sein muss, wo η_l eine Funktion von k_0 (reduzierter Wellenvektor für konstantes Potential) sein muss. Aus den erhaltenen zwei Gleichungsystemen kann a_{lmj} und b_{lmj} eliminiert werden und daraus folgt eine unendliche Determinantengleichung für $k_0(\mathbf{k})$ bzw. $E(\mathbf{k})$, aus der man jedoch schon unter Berücksichtigung von wenigen Zeilen und Kolonnen gute Näherungen für $E(\mathbf{k})$ erhält.

T. Neugebauer (Budapest).

Thermodynamics, Statistical Mechanics

de Groot, S. R. Sur la thermodynamique de quelques processus irréversibles. I. Corps simples. J. Phys. Radium (8) 8, 188–191 (1947).

Starting from the "flux-force" relations $J_i=\sum_k L_{ik}X_k$ and Onsager's relations $L_{ik}=L_{ki}$, the author develops the known formula $\Delta p/\Delta T=-Q^*/vT$ for a bipartite adiabatically isolated body, where $Q^*=Q-w$, $Q=L_{12}/L_{11}$, and $w=\mu+Ts$.

C. C. Torrance (Annapolis, Md.).

de Groot, S. R. Sur la thermodynamique de quelques processus irréversibles. II. Diffusion thermique et phénomènes connexes. J. Phys. Radium (8) 8, 193–200 (1947).

[Cf. the preceding review.] For a mixture of j components, the author represents the "flux" of matter by $J_i=\sum_k a_{ik}X_k+b_iX_u$ $(i=1,\cdots,j)$ and the "flux" of energy by $J_u=\sum_k b_k'X_k+cX_u$. Using Onsager's relations $a_{ik}=a_{ki}$ and $b_i=b_i'$, the author develops the theory of the Soret effect and related effects. *C. C. Torrance* (Annapolis, Md.).

Prins, J. A. A thermodynamical substitution group. Physica 13, 417–421 (1947).

The author gives a geometric diagram for the group of substitutions among P, V, T, S, and U, W, F, G.

C. C. Torrance (Annapolis, Md.).

Nath, Brij, and Auluck, F. C. Thermodynamics of a Fermi-Dirac gas obeying Born's modified quantum statistics. Proc. Nat. Inst. Sci. India 9, 257–264 (1943).

This paper lists power series expansions for E, S and G for a Fermi-Dirac gas on the basis of a finite momentum space using both relativistic and nonrelativistic assumptions.

C. C. Torrance (Annapolis, Md.).

Born, M., and Green, H. S. The kinetic basis of thermodynamics. Proc. Roy. Soc. London. Ser. A. 192, 166–180 (1948).

The general time-dependent perturbation problem is set up and solved exactly. The principle of conservation of energy is developed. Entropy is defined and shown to be additive for weakly coupled systems. A simple but rigorous proof of Boltzmann's law is given from which the thermodynamics of reversible processes may be inferred.

C. C. Torrance (Annapolis, Md.).

Vaĭnrib, E. A., and Spivak, G. V. A general method of performing calculations on statistical systems in a steady state. Doklady Akad. Nauk SSSR (N.S.) 59, 463–466 (1948). (Russian)

In the introduction to his "Statistical Thermodynamics" [Cambridge University Press, 1946; these Rev. 7, 540], Schrödinger states: "The distinguished role (in statistical thermodynamics) of the energy is, therefore, simply that it is a constant of the motion—the one that always exists, and, in general, the only one. The generalization to the case, that there are others besides (momenta, moments of momentum) is obvious; it has occasionally been contemplated, but in terrestrial, as opposed to astrophysical, thermodynamics it has hitherto not acquired any importance." The paper under review is an explicit formulation of this obvious generalization. The H-function expressed in a general form appropriate to Maxwell-Boltzmann, Fermi-Dirac or Bose-Einstein statistics is extremalized subject to the two usual scalar conditions (constancy of the number of particles, and of the energy of the system), plus five additional vector conditions (constancy of linear momentum, angular momentum, and of the fluxes of energy, linear and angular momentum). The distribution function is thus expressed in terms of seventeen Lagrangian undetermined multipliers, which are to be found by substitution into the above seventeen equations of condition. The general result is shown to reduce to previously published results in four special cases.

G. M. Volkoff (Vancouver, B. C.).

Landau, L. On the vibrations of the electronic plasma. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. 16, 574–586 (1946). (Russian)

Landau, L. On the vibrations of the electronic plasma. Acad. Sci. USSR. J. Phys. 10, 25–34 (1946).

Gentile, Giovanni. Le statistiche intermedie e le proprietà dell'elio liquido. Rend. Sem. Mat. Fis. Milano 15, 96–114 (1941).

Jost, Res. Eine Bemerkung über die Entropie in der Wellenmechanik. Helvetica Phys. Acta 20, 491–494 (1947).

In quantum statistical mechanics, the entropy S of an ensemble whose density matrix is P is given by the equation $S=-\text{trace } P\log P$ (taking the Boltzmann constant as unity). From this follows the well-known theorem that the entropy S of the combined ensemble (density $P=\alpha_1P_1+\alpha_2P_2$,

$\alpha_1+\alpha_2=1$, $\alpha_1>0$, $\alpha_2>0$), formed from the ensembles of densities P_i and entropies S_i ($i=1, 2$), satisfies the inequality $S\geqq\alpha_1S_1+\alpha_2S_2$, the equality corresponding to $P_1=P_2$. The author shows in this note that the inequality $S\leqq\alpha_1S_1+\alpha_2S_2-\alpha_1\log\alpha_1-\alpha_2\log\alpha_2$ holds, the equality corresponding to $P_1P_2=0$. Standard methods of analysis and wave mechanics are used. No applications are given.

B. O. Koopman (New York, N. Y.).

Davydov, B. Quantum mechanics and thermodynamic irreversibility. Acad. Sci. USSR. J. Phys. 11, 33–43 (1947).

In classical statistical mechanics the quasi-ergodic hypothesis enables one only to calculate the time averages for an isolated system. It is insufficient for determining the change in the state of a system during small time intervals and it does not lead, for instance, to the thermal conductivity equation. To obtain the laws of the variation with time the assumption of equal probabilities of all states compatible with the observed values of macroscopic quantities is necessary. The independence of the phases of gas molecules, leading to the usual collision law, is a special form of such an assumption. But being accepted for the instant t_1 the latter cannot be satisfied at other instants. A minimum at the time t_1 is thus obtained for the entropy instead of a monotonic increase. This is connected with the general contradiction between reversibility of classical mechanics and thermodynamic irreversibility.

The irreversibility of measurements in quantum mechanics leads to a permanent increase of entropy. Each measurement effects an averaging of phase differences between eigenfunctions corresponding to its different results. It enables one to prove the kinetic equation and the H-theorem, provided the measurements are repeated in short time intervals. At the instant of a measurement the entropy is not changed but each new measurement brings about the conditions of its subsequent increase.

Irreversibility of quantum mechanics is characterized by a "quantum entropy" $\sigma=-\text{spur}\,\rho\ln\rho$. It is not an observable quantity, thermodynamic entropy S being observable. It has been proved that $\Delta\sigma/\Delta t\geqq 0$ and $S\geqq\sigma$. Hence, if at the instant of a measurement $S=\sigma$ then later measurements give $\Delta S/\Delta t\geqq 0$ for any interval Δt between the measurements. The H-theorem is obtained under Pauli's assumption of equal probabilities of all states of any cell and of the independence of their phases. This assumption leads also to $S=\sigma$. According to the quasi-ergodic hypothesis it is satisfied nearly always owing to the former interaction of the system with the surrounding bodies. In the course of time it continues to hold. *Author's summary.*

Born, M., and Green, H. S. A general kinetic theory of liquids. I. The molecular distribution functions. Proc. Roy. Soc. London. Ser. A. 188, 10–18 (1946).

This paper is an outline of a general theory which has as an object the description of all the equilibrium and dynamical properties of liquids. The treatment is classical throughout and assumes that the molecules of the liquid interact with a simple potential energy function. A set of multiform distribution functions is defined and the equations of motion for these functions are derived. In addition, these functions are shown to obey equations of continuity. These equations are then shown to reduce to the Boltzmann equation in the theory of gases, for the limiting case when only binary collisions between particles play an important role. In addition, for the limiting case of equilibrium they are consistent with the results of ordinary statistical mechanics. In conclusion these equations are solved approximately for the so-called radial distribution function [Kirkwood and Boggs, J. Chem. Phys. 10, 394–402 (1942)] and a generalization of the equation of Kirkwood-Boggs is obtained.

J. M. Luttinger (Zurich).

Green, H. S. A general kinetic theory of liquids. II. Equilibrium properties. Proc. Roy. Soc. London. Ser. A. 189, 103–117 (1947).

This paper represents a continuation of the theory of part I [cf. the preceding review] and is entirely devoted to a study of the equilibrium properties of a system. The first half of the paper is concerned with a generalization of the Boltzmann H-theorem to the liquid phase, as well as a rederivation of the ordinary formula for the partition function of a liquid. In the second part the integral equation for the radial distribution function given in part I is further investigated and expansions are found which give the free energy of the liquid and gaseous phase. It is shown that the liquid phase is distinguished from the gaseous phase by the existence of real roots of a certain transcendental equation.

J. M. Luttinger (Zurich).

Born, M., and Green, H. S. A general kinetic theory of liquids. III. Dynamical properties. Proc. Roy. Soc. London. Ser. A. 190, 455–474 (1947).

The methods of part I [cf. the second preceding review] are now studied in detail for the dynamical behaviour of liquid. Generalized hydrodynamical equations are derived of which the fundamental equations of hydrodynamics are a special case; the more general equations describe the mean motion of clusters of molecules in the fluid. It is shown that the various quantities of interest (pressure tensor, energy-flux vector, coefficient of viscosity, coefficient of thermal conductivity) consist of two parts. One arises from the thermal motion, the other from the molecular forces. For a gas the former is dominant, for a liquid the latter. By an approximate solution of the rigorous equations, an expression for the viscosity is obtained which is in good agreement with the experimental facts. It is shown that in the dynamical case as well as in the equilibrium case analytical singularities arise which are related to the process of condensation. Finally, a full discussion is given for the one particle distribution function in nonuniform liquids and gases. *J. M. Luttinger* (Zurich).

Born, M., and Green, H. S. A general kinetic theory of liquids. IV. Quantum mechanics of fluids. Proc. Roy. Soc. London. Ser. A. 191, 168–181 (1947).

The considerations of the three papers reviewed above are now generalized to include quantum effects. The multiform distribution functions are replaced by matrices analogous to the density matrix of von Neumann. For these quantities equations of motion are derived which are very similar to those for the classical distribution functions. In the limit $h=0$ these equations go over into the previous classical ones. It is then shown that these generalized density functions may be used to derive the Boltzmann distribution law. Indications are given that the usual static notion of pressure and temperature need redefinition for the dynamical properties of liquids at low temperature. It is shown, however, that in terms of a new "dynamical" pressure and temperature the results of the former classical theory of thermal conductivity are identical with the quantum results. This fact is used to interpret some of the properties of He II.

J. M. Luttinger (Zurich).